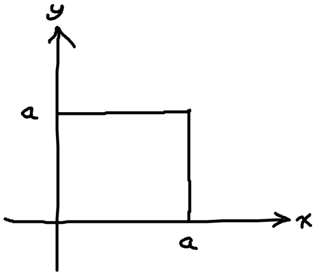
Vibrating membrane problems

2D Wave Equation

$$\nabla^2 \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c}\frac{\partial^2 u}{\partial t^2} \tag{1}$$

where $0 < x < a,\ 0 < y < a, t > 0$.



$u(x, y, t)$ satisfies $u = 0$ on the boundary with given initial conditions.
Plug $u(x, y, t) = X(x)Y(y)T(t)$ and its derivatives in the PDE.

$$\underbrace{\frac{X''}{X} + \frac{Y''}{Y}}_{\text{function of } x \text{ \& } y \text{ only}} = \underbrace{\frac{1}{c^2}\frac{T''}{T}}_{\text{function of } t \text{ only}} = k, \text{ constant.} \tag{2}$$

k is either

① $k > 0$
② $k = 0$
③ $k < 0$

Case ①.

$$\frac{1}{c^2}\frac{T''}{T} = k \tag{3}$$

$$T'' = kc^2 T \tag{4}$$

$$T'' - kc^2 T = 0 \tag{5}$$

$$e^{rt}(r^2 - kc^2) = 0 \tag{6}$$

$$\circledast\ r^2 = \underbrace{kc^2}_{+} \Rightarrow r = \pm c\sqrt{k} \Rightarrow T = Ae^{c\sqrt{k}t} + Be^{-c\sqrt{k}t}$$

If Case ② then $\circledast \Rightarrow\ T =$ (constant) which is a trivial solution which corresponds to a panel that is not vibrating at all.

Case ③. Take $k\ \ -\nu^2$.

$$\begin{aligned} T'' &= -\nu^2 c^2 T && (7)\\ e^{rt}(r^2+\nu^2c^2) &= 0 && (8)\\ &\Rightarrow r = \pm i\nu c && (9)\\ &\Rightarrow T(t) = A\cos{(\nu ct)} + B\sin{(\nu ct)} && (10) \end{aligned}$$

Nontrivial! Proceed with the xy-equation: separate again.

$$\underbrace{\frac{X''}{X}}_{\text{function of } x \text{ only}} = \underbrace{-\nu^2 - \frac{Y''}{Y}}_{\text{function of } y \text{ only}} = \text{constant, } l \tag{11}$$

Typical boundary conditions for this problem are holding the membrane down at the edges; $u = 0$ along the boundary.

Boundary conditions: $X(0) = X(a) = 0$

$$\begin{aligned} \frac{X''}{X} &= l \rightarrow X'' = lX && (12)\\ &\rightarrow e^{rx}(r^2 - l) = 0,\ r = \pm\sqrt{l} && (13) \end{aligned}$$

thus choose $l = -(\frac{n\pi}{a})^2$ where $n = 1, 2, \ldots$ so we get

$$X(x) = A\cos{(\frac{n\pi x}{a})} + B\sin{(\frac{n\pi x}{a})} \tag{14}$$

or combine using a common complex coefficient

$$X(x) = C\sin{(\frac{n\pi x}{a})} \qquad \text{such that } n = 1, 2, \ldots \tag{15}$$

Boundary conditions: $Y(0) = Y(a) = 0$

$$\begin{aligned} -\nu^2 - \frac{Y''}{Y} &= l && (16)\\ -\frac{Y''}{Y} &= l + \nu^2 && (17)\\ Y'' &= -(l+\nu^2)Y && (18) \end{aligned}$$

$$\text{Let } p = (-l - \nu^2):\ Y'' = pY. \tag{19}$$

$$\vdots$$

$$\text{from auxilliary equation } r = \pm\sqrt{p} \tag{20}$$

Why do we get to choose what l and p are? Particularly as $l = -(\frac{n\pi}{a})^2$ and $p = -(\frac{m\pi}{a})^2$?
Answer: Sturm-Liouville Theory. Functions as an orthogonal basis $L^2[a, b]\ \sigma(x)dx$.

Then the Y-solution is

$$Y(y) = D\sin(\frac{m\pi y}{a}) \qquad m = 1, 2, \ldots \tag{21}$$

Remember what p was designated to represent? Run with it.

$$p = \nu^2 - l \tag{22}$$

$$\to -p - l = \nu^2 = (\frac{m\pi}{a})^2 + (\frac{n\pi}{a})^2 \Rightarrow \nu_{mn} \equiv \frac{\pi}{a}\sqrt{m^2 + n^2} \tag{23}$$

which is a nice looking eigenvalue to provide a (generally) periodic solution.

ν_{mn} goes into the T-equation from the nontrivial Case ③:

$$T(t) = A\cos(\frac{\pi c}{a}\sqrt{m^2 + n^2}t) + B\sin(\frac{\pi c}{a}\sqrt{m^2 + n^2}t) \tag{24}$$

Assemble the solution:

$$\nu_{mn} = \sin(\frac{m\pi x}{a})\sin(\frac{n\pi x}{a}) \cdot \{A_{mn}\cos\nu_{mn}ct + B_{mn}\sin(\nu_{mn}ct) \tag{25}$$

Each harmonic mode can be superposed:

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn}(x, y, t) \tag{26}$$

so as to satisfy initial conditions of fixed sides. Typically, we are given something along the lines of a displacement at particular time t ($t = 0 \Leftrightarrow$ initial position).

$$u(x, y, 0) = \alpha(x, y) \Rightarrow \alpha(x, y) = \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} A_{nm}\sin(\frac{n\pi x}{a})\right)\sin(\frac{m\pi y}{a}) \tag{27}$$

$$\frac{\partial u}{\partial t}(x, y, 0) = \beta(x, y) \qquad \text{initial velocity} \tag{28}$$

This can be done with Fourier Series.

For <u>fixed x</u>, $\sum_{n=1}^{\infty} A_{nm}\sin(\frac{n\pi x}{a})$ depends only on m, and turn out to be the Fourier coefficients of the Fourier sine series of y of $\alpha(x, y)$ over $o < y < a$. Thus:

$$\sum_{n=1}^{\infty} A_{nm}\sin\frac{n\pi x}{a} = \frac{2}{a}\int_0^a \alpha(x, y)\sin\frac{m\pi y}{a}dy \tag{29}$$

Notice that the left is the Fourier sine series of the right-hand side, in x.

To make a more general display of the double Fourier Series result consider when it is the case that $0 < x < L$ and $0 < y < H$.

$$A_{nm} = \frac{2}{L}\int_0^L \left[\frac{2}{H}\int_0^H \alpha(x,y)\sin\frac{m\pi y}{H}dy\right]\sin\frac{n\pi x}{L}dx \tag{30}$$

$$= \frac{4L}{H}\int_0^L\int_0^H \alpha(x,y)\sin\frac{m\pi y}{H}\sin\frac{n\pi x}{L}dy\ dx \tag{31}$$

From initial condition $\frac{\partial u}{\partial t}(x,y,0) = \beta(x,y)$ treated by similar methods we obtain

$$c\nu_{nm}B_{nm} = \frac{4}{LH}\int_0^L\int_0^H \sin\frac{m\pi y}{H}\sin\frac{n\pi x}{L}\beta(x,y)dy\ dx \tag{32}$$

Summary

$$\nabla^2 \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c}\frac{\partial^2 u}{\partial t^2} \tag{33}$$

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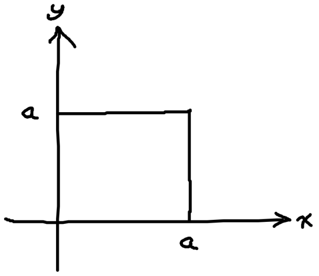

→ Harmonic time dependent vibrations on fixed ends with initial displacement (α) and initial velocity (β).

Separable solution:

$$u(x,y,t) = \sin\frac{m\pi y}{a}\sin\frac{n\pi x}{a}\cdot\{A_{mn}\cos(\nu_{mn}ct) + B_{mn}\sin(\nu_{mn}ct)\} \tag{34}$$

where

$$\nu_{mn} = \frac{\pi}{a}\sqrt{m^2+n^2} \tag{35}$$

$$A_{mn} = \left(\frac{2}{a}\right)^2\int_0^a\int_0^a \alpha\sin\frac{m\pi y}{a}\sin\frac{n\pi x}{a}dy\ dx \tag{36}$$

$$B_{mn} = \left(\frac{2}{a}\right)^2\int_0^a\int_0^a \beta\sin\frac{m\pi y}{a}\sin\frac{n\pi x}{a}dy\ dx \tag{37}$$