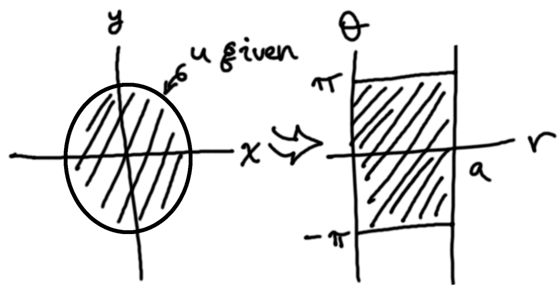
Again we will develop a solution to the wave equation for a vibrating membrane except in polar coordinates.

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad u(r,\theta,t) = u \tag{1}$$

But first we need $\nabla^2 u$ in plane polars

$$x = r\cos\theta,\ \ y = r\sin\theta. \tag{2}$$

Disc: $x^2 + y^2 \leq a^2$



We will handle each of the terms from (1) separately i.e., $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$.

First consider taking the partial derivative of u with respect to x given as a function in terms of new variables r and θ whereby we use the chain rule, namely:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} \tag{3}$$

To compute these various derivatives, consider the change of variables equations from rectangular x, y coordinates to polar r, θ coordinates.

First differentiate $r^2 = x^2 + y^2$ with respect to x:

$$\begin{aligned} \frac{\partial}{\partial x}\left(r^2 = x^2 + y^2\right) &\Rightarrow 2r\frac{\partial r}{\partial x} = 2x \\ &\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta. \end{aligned} \tag{4}$$

Next, differentiate $\tan\theta = y/x$ in terms of x with respect to θ:

$$\begin{aligned} \frac{\partial}{\partial \theta}\left(tan\theta = y/x\right) &\Rightarrow \sec^2\frac{\partial \theta}{\partial x} = -\frac{y}{x^2} \\ &\Rightarrow \frac{\partial \theta}{\partial x} = \frac{-y}{x^2}\cos^2\theta = \frac{-\sin\theta}{r} \end{aligned} \tag{5}$$

Using (4) and (5) we may now re-write (3) as

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\cos\theta - \frac{\partial u}{\partial \theta}\sin(\theta/r). \tag{6}$$

Second consider taking the partial derivative of u with respect to y given as a function in terms of new variables r and θ whereby we use the chain rule again, namely:

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} \tag{7}$$

Again, to compute these various derivatives, consider the change of variables equations from rectangular x, y coordinates to polar r, θ coordinates, but instead this time differentiate $r^2 = x^2 + y^2$ with respect to y:

$$\begin{aligned} \frac{\partial}{\partial y}\left(r^2 = x^2 + y^2\right) &\Rightarrow 2r\frac{\partial r}{\partial y} = 2y \\ &\Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta. \end{aligned} \tag{8}$$

Once again, differentiate $\tan\theta = y/x$ in terms of y with respect to θ:

$$\begin{aligned} \frac{\partial}{\partial \theta}\left(tan\theta = y/x\right) &\Rightarrow \sec^2\frac{\partial \theta}{\partial y} = -\frac{1}{x} \\ &\Rightarrow \frac{\partial \theta}{\partial y} = \frac{1}{x}\cos^2\theta = \frac{\cos\theta}{r} \end{aligned} \tag{9}$$

Using (8) and (9) we may now write (7) as

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\sin\theta + \frac{\partial u}{\partial \theta}\cos\left(\theta/r\right). \tag{10}$$

We will continue by differentiating (6) again with respect to x, thus:

$$\begin{aligned} &\frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\cos\theta - \frac{\partial u}{\partial \theta}\sin\left(\theta/r\right)\Big) \\ &\frac{\partial^2 u}{\partial x^2} = (\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta})(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}) \\ &\quad = \cos^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} - \frac{2}{r}\sin\theta\cos\theta\frac{\partial^2 u}{\partial\theta\partial r} + \frac{2}{r}\sin\theta\cos\theta\frac{\partial u}{\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial^2 u}{\partial\theta^2} \end{aligned} \tag{11}$$

Similarly, continue by differentiating (9) again with respect to y, thus:

$$\begin{aligned} &\frac{\partial}{\partial y}\Big(\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\sin\theta + \frac{\partial u}{\partial \theta}\cos\left(\theta/r\right)\Big) \\ &\frac{\partial^2 u}{\partial y^2} = (\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta})(\sin\theta\frac{\partial u}{\partial r} + \cos\theta\frac{\partial u}{\partial \theta}) \\ &\quad = \sin^2\theta\frac{\partial^2 u}{\partial r^2} + \sin\theta\cos\theta\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} - \frac{\cos\theta\sin\theta}{r}\frac{\partial u}{\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial^2 u}{\partial\theta^2} \end{aligned} \tag{12}$$

Combining (11) and (12) gives

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \tag{13}$$

In the case of the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u \tag{14}$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}; \qquad \text{s.t. } x^2 + y^2 \leq a \tag{15}$$

the solution will be of the form $u(r, \theta, t)$.

Assume zero displacement at the circular boundary: $u(a, \theta, t) = 0$.

$u = u(r, \theta, t) = P \cdot T(t)$ where $P = P(\theta, r) = \Theta(\theta)R(r)$.

Note that if $Q = Q(x,y) = X(x)Y(y)$ then the rectangular vibrating membrane is solved by $u = u(x,y,t) = Q \cdot T(t)$.

So as before, —T-Equation— $T'' = -\nu^2 c^2 T \Rightarrow \ \dots \ T(t) = A\cos(\nu ct) + B\sin(\nu ct)$

—R,Θ Equations—

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$u(r,\theta) = \Theta(\theta)R(r)$, PDE: $\Theta(\theta)R''(r) + \frac{1}{r}\Theta(\theta)R'(r) + \frac{1}{r^2}\Theta''(\theta)R(r) = -\nu^2\Theta(\theta)R(r)$

$u_r'' = \Theta(\theta)R''(r)$
$u_r' = \Theta(\theta)R'(r)$
$u_\theta'' = \Theta''(\theta)R(r)$

$$\Rightarrow \frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} + \frac{\Theta''(\theta)}{r^2\Theta(\theta)} = -\nu^2$$

$$\Rightarrow \frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} + \nu^2 = -\frac{\Theta''(\theta)}{r^2\Theta(\theta)}$$

→ Multiply through by r^2:

$$\Rightarrow r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} + \nu^2 r^2 = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \mu, \text{ constant.} \tag{16}$$

Notice that the LHS of (16) is a function of r only and the RHS is a function of θ only.
Thus, we must solve the following:

$$\Theta''(\theta) = -\mu\Theta(\theta)$$

—θ-equation—

$$r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} + \nu^2 r^2 = \mu$$

—r-equation—

$$r\left\{r\frac{R''(r)}{R(r)} + \frac{R'(r)}{R(r)}\right\} + \nu^2 r^2 = \mu \qquad -\pi < \theta < \pi;\ 0 < r < a$$

$$\frac{r}{R(r)}\{rR''(r) + R'(r)\} + \nu^2 r^2 = \mu \tag{17}$$

$$\frac{r}{R(r)}\frac{d}{dr}\{rR'(r)\} + \nu^2 r^2 = \mu \tag{18}$$

$$\Rightarrow r\frac{d}{dr}\{rR'(r)\} + \nu^2 r^2 R(r) = \mu R(r) \tag{19}$$

$$\Rightarrow r\frac{d}{dr}\{rR'(r)\} + (\nu^2 r^2 - \mu)R(r) = 0 \text{ r-equation} \tag{20}$$

→ Divide by r to put in singular Sturm-Liouville form:

$$\frac{d}{dr}\{rR'(r)\} + (\nu^2 r - \frac{\mu}{r})R(r) = 0$$

Reminder: Regular Sturm-Liouville form is

$$(pf')' + qf + \lambda\sigma f = 0$$

It is customary to impose periodic boundary conditions on θ:

$$\Theta(-\pi) = \Theta(\pi)$$
$$\Theta'(-\pi) = \Theta'(\pi)$$

Determine the nature of eigenvalue μ by solving the —θ-equation—.

$$\Theta''(\theta) = -\mu\Theta(\theta)$$
$$e^{r\theta}(r^2 + \mu) = 0$$
$$r = \pm i\sqrt{\mu}$$
$$\rightarrow \Theta(\theta) = c_1 \cos(\sqrt{\mu}\theta) + c_2 \sin(\sqrt{\mu}\theta).$$

The θ problem shows us that we have two eigenfunctions from the periodic boundary conditions.

$$\Theta(\theta) = \cos(m\theta) \text{ and } \Theta(\theta) = \sin(m\theta) \text{ for } \mu = m^2,\ m = 0,\ 1,\ 2,\ \ldots$$

—r-equation— is special, and is called Bessel's equation, obtained by:

$$r^2R''(r) + rR'(r) + (\nu^2r^2 - m^2)R(r) = 0$$

which is said to be in singular Sturm-Liouville form.

Make the change to $x = \nu r \Leftrightarrow \frac{x}{\nu} = r$. Start by dividing through by ν^2 before substituting to get something along the lines of Bessel's Equation of order m:

$$x^2w''(x) + xw'(x) + (x^2 - m^2)w(x) = 0$$

Some things to remember about Bessel's equation:

- The point $z = 0$ is singular
- $f \approx 1$ and $f \approx \ln z$ for $m = 0$

i.e., a well-behaved and a singular solution.

Use the method of Frobenius to solve Bessel's equation; seek solutions of the form of an infinite power series. Do not use $w(x) = \sum_{n=0}^{\infty} a_n x^n$ but rather the more general

$$w(x) = \sum_{m=0}^{\infty} c_m x^{m+\alpha} = x^\alpha \sum_{m=0}^{\infty} c_m x^m$$

$$w'(x) = \sum_{m=0}^{\infty} c_m x^{m+\alpha-1}(m+\alpha)$$

$$w''(x) = \sum_{m=0}^{\infty} c_m x^{m+\alpha-2}(m+\alpha)(m+\alpha-1)$$

$$x^2w''(x) = \sum_{m=0}^{\infty} c_m x^{m+\alpha}(m+\alpha)(m+\alpha-1)$$

$$xw'(x) = \sum_{m=0}^{\infty} c_m x^{m+\alpha}(m+\alpha)$$

$$-n^2w(x) = \sum_{m=0}^{\infty} c_m x^{m+\alpha}(-n^2)$$

$$x^2 w(x) = \sum_{m=0}^{\infty} c_m x^{m+\alpha+2} = \sum_{m=2}^{\infty} c_{m-2} x^{m+\alpha}$$

Combine to yield

$$\sum_{m=0}^{\infty} c_m x^{m+\alpha} \left\{ (m+\alpha)(m+\alpha-1) + (m+\alpha) - n^2 \right\} + \sum_{m=2}^{\infty} c_{m-2} x^{m+\alpha} = 0$$
$$\sum_{m=0}^{\infty} c_m x^{m+\alpha} \left\{ (m+\alpha) - n^2 \right\} + \sum_{m=2}^{\infty} c_{m-2} x^{m+\alpha} = 0$$

Note that $c_0 \neq 0$.

In order for our equation to be true for a set of x-values, the coefficients multiplying the x-power must equal 0.

—$m = 0$— (x^{α}) : $c_0\{\alpha^2 - n^2\} = 0 \ \leftarrow$ "Indicial equation"
—$m = 1$— $(x^{1+\alpha})$: $c_1\{(\alpha+1)^2 - n^2\} = 0$
—$m \geq 2$— $(x^{m+\alpha})$: $c_m\{(m+\alpha)^2 - n^2\} + c_{m-2} = 0$. Note that this works for all $m = 2, 3, 4, \ \ldots$

Since $c_0 \neq 0$ then the —$m = 0$— case gives

$$\alpha^2 - n^2 = 0 \ \Rightarrow \ \alpha \pm n$$

Note beforehand that $\alpha = -n$ does not give a solution of Bessel's equation, or at least we're told that, so ...

Plug into —$m = 1$—

$$c_1\{(n+1)^2 - n^2\} = 0$$
$$c_1(2n+1) = 0 \text{ thus } c_1 = 0 \qquad (21)$$

Note in (21) that $(2n+1) \neq 0$.

Plug into —$m \geq 2$—

$$c_m\{(m+n)^2 - n^2\} + c_{m-2} = c_m\{m^2 + 2mn\} + c_{m-2}$$

Thus $c_m = -\frac{c_{m-2}}{m(2n+m)}$, so we will inspect both even and odd values of $m \geq 2$.
Case: $m = 2l+1$, $l = 1, \ 2, \ \ldots$

$$c_{2l+1} = -\frac{c_{2l-3}}{(2l+1)(2n+2l+1)}$$
$$c_{2l-1} = -\frac{c_{2l-3}}{(2l-1)(2n+2l-1)}$$

Let $\alpha = (2l-1)(2n+2l-1)$ and $\beta = (2l-1)(2n+2l+1)$, thus $c_{2l+1} = \frac{1}{\alpha\beta} c_{2l-3}$.

$\Rightarrow$ Extend the process recursively back to $c_{2l+1} = f(m,n)c_1 = 0$.

Case: $m = 2l$, $l = 1, \ 2, \ 3, \ \ldots$

$$c_{2l} = -\frac{c_{2l-2}}{(2l)(2l+2n)} = \frac{-c_{2l-2}}{4l(l+n)}$$

$$c_{2l-2} = -\frac{c_{2l-4}}{(2l-2)(2n+2l-2)} = \frac{-c_{2l-4}}{2(l-1)(n+l-1)}$$

$$c_{2l-4} = -\frac{c_{2l-6}}{(2l-4)(2n+2l-4)} = \frac{-c_{2l-6}}{2(l-2)(n+l-2)}$$

$$c_{2l} = -\frac{c_{2l-6}}{(2l)(2l-2)(2l-4)(2l+2n)(2l+2n-2)}$$

$\Rightarrow$ Extend the process recursively back to c_0.

$$c_{2l} = \frac{(-1)^l \cdot c_0}{[4l \cdot (l+n)][4(l-1)(l+n-1)]\ldots[4(n+1)]}$$

Notice in each pair of square brackets that there is $4 = 2^2$, l-times, there is also $(l+n)\cdot(l+n-1)\cdot\ldots(n+1)$, and there is also $l \cdot (l-1)\cdot(l-2)\ldots$, thus

$$c_{2l} = \frac{(-1)^l n!}{2^{2l} l!(l+n)!} c_0,$$

and we have

$$\sum_{m=0}^{\infty} c_m x^{m+n} = \sum_{m=1,\ 3,\ 5,\ \ldots}^{\infty} c_m x^{m+n} + \sum_{m=0,\ 2,\ 4,\ \ldots}^{\infty} c_m x^{m+n} \tag{22}$$

However, letting $m = 2l$ allows us to be rid of the first sum on the RHS of (23).

Let us write

$$y = \sum_{l=0}^{\infty} c_0 \frac{(-1)^l n!}{2^{2l} l!(l+n)!} x^{2l+n}$$

Define the Bessel function (of the first kind):

$$J_n(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l+n} l!(l+n)!} x^{2l+n}$$

Thus

$$y = c_0 2^n n! J_n(x) = \tilde{c_0} J_n(x) \tag{23}$$

where it means in (23) above that $c_0 2^n n!$ is a constant that has been re-written as, say $\tilde{c_0}$.

Ex. Heat conduction in a semi-infinte solid. First pick an integral transform method to use:

- Laplace transform in t
- Laplace transform in x
- Fourier sine transform in x which leads to a 1^{st} order linear, non-homogeneous differential equation that could possibly be solved with integrating factors.