

The Work of a Reader to Fill In The Gaps Of Riemann's *Über die Hypothesen Welche der Geometrie zu Grunde Liegen* as an Exercise

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1 Introduction

The result worked out here is to derive a metric for a constant Gaussian curvature n -dimensional manifold with a locally Euclidean metric that is scaled by a conformal (i.e., angle preserving) factor. This amounts to finding a solution of a form of the Liouville equation, a partial differential equation. The motivation is to demonstrate a solution of the Liouville equation for the Riemannian metric of a Riemannian manifold given constant Gaussian curvature K . The particular form of the Liouville equation here displays a solution that will resemble the stereographic projection formulae and connect to the result in Part II. §4 of Riemann's 1854 Habilitationsschrift dissertation (hereafter "*Über die Hypothesen*") [1].

The result in Part II. §4 of Riemann's Habilitation thesis is a classification result for Riemannian manifolds, namely for constant Gaussian curvature K when $K > 0$ it is called spherical curvature, $K = 0$ corresponds to Euclidean or flat curvature, and $K < 0$ is called hyperbolic curvature.

2 Background

The goal is to show that

$$K = \frac{R_{1212}}{g} \Rightarrow \frac{1}{1 + \frac{1}{4}\alpha \sum(x^2)} \sqrt{\sum dx^2} \quad (1)$$

Roughly speaking, the left side of (1) is a statement of Gaussian curvature in terms of a component of the Riemannian curvature tensor and a metric g . The right side of (1) is the result of Riemann's 1854 habilitation thesis. The only symbolic math notation expressed in the entirety of the dozen or so pages of Riemann's *Über die Hypothesen* as it appears in [1] is the right-hand side of (1). It is implied that the expression on the right-hand side of (1) is an expression equal to ds or the notion of infinitesimal arc length on the surface of the manifold. Essentially, (1) implies the gap that was left by Riemann for the readers of his habilitation thesis to fill in as an exercise.

The more abstract goal is to better understand how to use metrics to determine lengths, angles, and areas, or in other words how to use "measure-relations" and "magnitude-relations" as they perhaps were to Riemann, on n -manifolds of constant Gaussian (or sometimes *sectional*) curvature.

2.1 The Problem in Plain Terms

Consider rulers whose lengths are greater than the upper bound of any ruler too small to meaningfully measure lengths on the manifold of interest, but also less than being so long that there would be noticeable deformation to the rulers by stretching or bending as the rulers are moved to different positions throughout the manifold. The rulers of these specified lengths are likened to the dx^i used to describe the metric being

sought. The metric sought is thus a relation of such given rulers that can be used to provide a measurement of notions such as length, area, or curvature that doesn't violate Euclid's fifth postulate nor get confused between situations such as being on a cylinder or a flat plane depending how the situation is interpreted.

2.2 Definitions

First, here are some definitions of concepts from smooth manifolds and modern (i.e., after circa 1950) Riemannian geometry related to what is discussed here. Consider the notion of smooth maps in the context of Euclidean spaces. If U and V are open subsets of Euclidean spaces \mathbb{R}^n and \mathbb{R}^m , respectively, a function $F : U \rightarrow V$ is said to be **smooth** (or C^∞ , or **infinitely differentiable**) if each of its component functions has continuous partial derivatives of all orders.

Topological manifolds are regarded as topological spaces, M say, that are Hausdorff spaces (for every pair of distinct points $p, q \in M$, there are disjoint open subsets $U, V \subseteq M$ such that $p \in U$ and $q \in V$), have a countable basis for the underlying topology, and are locally n -dimensionally Euclidean (each point of M has a neighborhood that is homeomorphic to an open subset of \mathbb{R}^n , or an open ball in \mathbb{R}^n , or \mathbb{R}^n itself).

A **coordinate chart** or just a **chart** on M is a pair (U, ϕ) , where U is an open subset of M and $\phi : U \rightarrow \widehat{U}$ is a homeomorphism from U to an open subset $\widehat{U} = \phi(U) \subseteq \mathbb{R}^n$. An **atlas for M** is a collection of charts whose domains cover M .

If (U, ϕ) , (V, ψ) are two charts such that $U \cap V \neq \emptyset$, the composite map $\psi \circ \phi^{-1} : \phi(U \cap V) \rightarrow \psi(U \cap V)$ is called the **transition map from ϕ to ψ** . Two charts are said to be **smoothly compatible** if either $U \cap V = \emptyset$ or the transition map $\psi \circ \phi^{-1}$ is a diffeomorphism. An atlas \mathcal{A} is called a **smooth atlas** if any two charts in \mathcal{A} are smoothly compatible with each other. A smooth atlas \mathcal{A} on M is **maximal** if it is not properly contained in any larger smooth atlas.

If M is a topological manifold, a **smooth structure on M** is a maximal smooth atlas. A **smooth manifold** is a pair (M, \mathcal{A}) , where M is a topological manifold and \mathcal{A} is a smooth structure on M . **Riemannian manifolds** are smooth manifolds equipped with **Riemannian metrics** (which are simply smoothly varying choices of inner products on tangent spaces), which allow for the measurement of geometric quantities such as distances and angles. The notion of curvature and its relation to topology is the central unifying theme of current Riemannian geometry research.

The mathematically rigorous generalization of the notion of geometric objects that intersect other geometric objects at only one point might begin with tangent lines to plane curves. These curves get represented as functions and the tangent lines are expressed as derivatives of these functions. At a given point p on a curve there is a tangent γ' along the curve at that point. Many circles can be coincidentally tangent to the tangent γ along the curve at that point. The circle among these whose own tangent at p is the same as that of γ is called the **osculating circle**. Curvature is thus $\kappa = 1/R$ where R is the radius of the osculating circle.

The next step is to develop this notion of curvatures for curved 2-dimensional surfaces in 3-dimensional space. This can be done with what are called the **principal curvatures**. Kreyszig's Differential Geometry defines principal curvatures starting with **normal curvatures** which are the curvatures of the curves intersecting normal sections of the surface of interest at a particular point. The curvature in the directions for which this normal curvature becomes extreme are the principal curvatures.

It turns out that principal curvatures are not *intrinsic*, which is a property paramount to Riemannian geometry. Properties of surfaces in \mathbb{R}^3 are called **intrinsic** if they are preserved by *isometries* (maps from one surface to another that preserve lengths of curves). Roughly speaking this means intrinsic properties are those that can be gleaned from a microscopic local perspective rather than a macroscopic global perspective. To see this, consider the normal sections and principal curvatures of a plane versus those of a cylinder.

2.3 Historical Context

In spite of principal curvatures not being intrinsic, in 1827 Gauss discovered what is now known as the *Theorema Egregium*, namely that the so-called **Gaussian curvature**,

$$K = \kappa_1 \kappa_2 \quad (2)$$

or the product of the principal curvatures, is intrinsic by showing that it depends only on what is called the first fundamental form.

Consider a component of the first fundamental form,

$$g_{\alpha\beta} = \mathbf{x}_\alpha \cdot \mathbf{x}_\beta \quad (3)$$

where the \mathbf{x} are tangent vectors to a surface and where α and β are indices of principal directions on that surface. More precisely, let t parameterize a function $\mathbf{u}(\mathbf{x}_i)$ representing position on the surface and $\alpha, \beta \in \{i\}$. For the moment limit to just 2 dimensions and say that there is only \mathbf{x}_1 and \mathbf{x}_2 . In this case the first fundamental form may be presented as

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2 \quad (4)$$

where E , F , and G are metric coefficients. Stretching amounts to conformal dilation of E , F , and/or G and deforming K amounts to bending. The discriminant of the first fundamental form then would be

$$g \equiv \det(g_{\alpha\beta}) = g_{11}g_{22} - (g_{12})^2 \quad (5)$$

Consider along the lines of (3) a different dot product of vectors at a point on a surface referred to as the second fundamental form:

$$b_{\alpha\beta} = \mathbf{x}_{ab} \cdot \mathbf{n} = -\mathbf{x}_\alpha \cdot \mathbf{n}_\beta \quad (6)$$

$$b_{\alpha\beta} = \frac{|\mathbf{x}_1 \times \mathbf{x}_2 \cdot \mathbf{x}_{\alpha\beta}|}{\sqrt{g}} \quad (7)$$

where $\mathbf{x}_{ab} = \frac{\partial^2 \mathbf{u}}{\partial \mathbf{u}^\alpha \partial \mathbf{u}^\beta}$ and \mathbf{n} is the unit normal vector on the surface $\mathbf{n} = \frac{\mathbf{x}_1 \times \mathbf{x}_2}{|\mathbf{x}_1 \times \mathbf{x}_2|}$. Similar to (5) there is also the corresponding discriminant of the second fundamental form:

$$b \equiv \det(b_{\alpha\beta}) = b_{11}b_{22} - (b_{12})^2 \quad (8)$$

By means of (5) and (8), (2) may be written as

$$K = \frac{b}{g} \quad (9)$$

However (7) shows that b depends on the first fundamental form only, therefore K depends on the first fundamental form only. The explicit representation of Gaussian curvature K only in terms of $g_{\alpha\beta}$ is sometimes called the *equation of Gauss*:

$$K = \frac{1}{g^2} \begin{vmatrix} g_{11} & g_{12} & \frac{\partial g_{12}}{\partial u^2} - \frac{1}{2} \frac{\partial g_{22}}{\partial u^1} \\ g_{12} & g_{22} & \frac{1}{2} \frac{\partial g_{22}}{\partial u^2} \\ \frac{1}{2} \frac{\partial g_{11}}{\partial u^1} & \left(\frac{\partial g_{12}}{\partial u^1} - \frac{1}{2} \frac{\partial g_{11}}{\partial u^2} \right) & \left(-\frac{1}{2} \frac{\partial^2 g_{11}}{(\partial u^2)^2} + \frac{\partial^2 g_{12}}{\partial u^1 \partial u^2} - \frac{1}{2} \frac{\partial^2 g_{22}}{(\partial u^1)^2} \right) \end{vmatrix} - \frac{1}{g^2} \begin{vmatrix} g_{11} & g_{12} & \frac{1}{2} \frac{\partial g_{11}}{\partial u^2} \\ g_{12} & g_{22} & \frac{1}{2} \frac{\partial g_{22}}{\partial u^1} \\ \frac{1}{2} \frac{\partial g_{11}}{\partial u^2} & \frac{1}{2} \frac{\partial g_{22}}{\partial u^1} & 0 \end{vmatrix} \quad (10)$$

This is effectively what Gauss' Theorema Egregium states:

Theorem 1 (*Theorema Egregium, Gauss*). *The Gaussian curvature K of a surface is independent of the second fundamental form but depends only on the coefficients $g_{\alpha\beta}$ of the first fundamental form (and their first and second derivatives).*

Riemann's *Über die Hypothesen* effectively extends Gauss' differential geometry of surfaces to n -manifolds while simultaneously advancing a classification result of these n -manifolds in the case of constant Gaussian curvature. Riemann accomplished this in *Über die Hypothesen* by laying the conceptual groundwork of Riemannian metrics on Riemannian manifolds and the associated Riemann curvature tensor.

This is not the kind of terminology that Riemann himself used in 1854 to advance these ideas for the first time, however. Riemann's habilitation thesis often refers to "measure-relations" and "magnitude-relations". So-called "magnitude-relations" in discrete spaces are obtained by counting. "Magnitude-relations" in continuous spaces, on the other hand, are obtained by measuring and comparing. On the one hand, the Pythagorean Theorem provides a way of measuring lengths (i.e., magnitudes) on flat planes. On the other hand, Euclid's fifth postulate on the eventual intersection of parallel lines was troublesome for the global geometry of surfaces in the mid-1800s. Riemann surmounted this by conceptually advancing upon his advisor Gauss' Theorema Egregium. In fact, the second fundamental form (7) that was shown to be not-so-fundamental after all to Gaussian curvature K can now be represented as merely a component of the Riemannian curvature tensor divided by the discriminant of the first fundamental form:

$$K = \frac{R_{1212}}{g} \quad (11)$$

An approach to determining an intrinsic metric or "measure-relation" on n -manifolds of constant Gaussian curvature might perhaps be to scale the simplest feasible locally Euclidean metric in a way that preserves angles. In other words, an approach to an intrinsic metric on a surface that can give insight into its curvature might be to conformally scale the infinitesimal n -dimensional Euclidean metric to obtain an infinitesimal $n - 1$ -dimensional metric on a surface in that space.

In Part II §4 of *Über die Hypothesen* under the header "Flat manifoldness may be treated as a special case of manifoldness with constant curvature", Riemann announces that he has more or less just succeeded with that approach. The result he presents at the end of §4 in Part II is

$$ds = \frac{\sqrt{\sum (dx^i)^2}}{1 + \frac{\alpha}{4} \sum (x^i)^2}, \quad (12)$$

where $\alpha = 4K$.

2.4 Mathematically Framing the Problem

The Riemannian metric is the sort of relation being sought, which together with a smooth manifold makes a Riemannian manifold. This Riemannian metric can pose a Riemannian curvature tensor which then can be used to compute curvature. The tensor calculations of Riemannian geometry were not laid out by Riemann himself but rather afterwards by Gregorio Ricci-Curbastro and Tullio Levi-Civita. Ricci-Curbastro introduced the modern formation of the Riemannian curvature tensor. Levi-Civita would introduce, among other things in this subject of mathematics, what is now known as Ricci curvature. Ricci Curvature R_{ij} is a tensor that is a trace resulting from a tensor contraction of the Riemannian curvature tensor R_{ikj}^k . Levi-Civita introduced this to simplify the calculation of the full Riemannian curvature tensor.

Thus the surfaces of non-zero constant curvature can be explored through the Riemannian curvature tensor of Riemannian manifolds with Riemannian metrics that are the Euclidean metric scaled by a constant conformal factor. First, the Riemannian curvature tensor is contracted to a statement of Ricci curvature. Then, the statement of conformal transformation of the Euclidean metric leads to the Ricci curvature involving a

PDE of the Laplacian in the form of Liouville's equation.

To illustrate this, consider for example a specific instance of the metric on an $n - 1$ dimensional surface embedded in n -dimensional space, namely a 2D surface embedded in 3D Euclidean space. The case of computing curvature in n -dimensional space for $n = 3$ would require defining $n - 1 = 2$ principal coordinate directions on the 2D surface of interest. In other words, a manifold embedded in n -dimensional ambient space takes at most $n - 1$ coordinates to unambiguously define position upon the manifold. The example 2D surface might thus be said to be parametrized by (u, v) . The metric would be given by $ds^2 = g_{ij}dx^i dx^j$. This would just be $ds^2 = du^2 + dv^2$ in the case of a flat plane. For surfaces of non-zero curvature, scaling the flat metric by a general conformal factor $ds^2 = e^{2u} \sum (dx^i)^2 = \lambda \sum (dx^i)^2$ will be considered.

Liouville's Equation appears in the study of isothermal coordinates so far as the independent variables are the coordinates being mapped (see §62. *Conformal mapping of surfaces into a plane* from [2]), while the exact solution function f can be described as the conformal factor with respect to the flat Euclidean metric. Its general form is

$$\Delta_0 \log f = -K f^2 \tag{13}$$

where $\Delta_0 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Note that the more general Liouville Equation may be written in terms of the Laplace-Beltrami operator $\Delta_{LB} = \frac{1}{f^2} \Delta_0$ so that we obtain $\Delta_{LB} \log f = -K$. The Laplace-Beltrami operator is the topic of Kreyszig §75. *Differential parameters of Beltrami*. [2]. The Laplace-Beltrami operator was not included in the scope of Lee's Intro to Riemannian Manifolds book [4], but it is indeed covered as a sub-topic within the context of Hodge Theory in Peter Petersen's Riemannian Geometry book [5].

Note the following result from chapter 7 "Curvature" under the heading "Curvatures of Conformally Related Metrics" in Lee's Intro to Riemannian Manifolds book we find on page 217 [4]. Specifically, (16) will come up again below at (94):

Theorem 2 (Conformal Transformation of the Curvature). *Let g be a Riemannian or pseudo-Riemannian metric on an n -manifold M with or without boundary, $f \in C^\infty(M)$, and $\tilde{g} = e^{2f}g$. In the Riemannian case, the curvature tensors of \tilde{g} (represented with tildes) are related to those of g by the following formulas:*

$$\widetilde{Rm} = e^{2f}(Rm - (\nabla^2 f) \otimes g + (df \otimes df) \otimes g - \frac{1}{2}|df|_g^2(g \otimes g)), \tag{14}$$

$$\widetilde{Rc} = Rc - (n - 2)(\nabla^2 f) + (n - 2)(df \otimes df) - (\Delta f + (n - 2)|df|_g^2)g, \tag{15}$$

$$\widetilde{S} = e^{-2f}(S - 2(n - 1)\Delta f - (n - 1)(n - 2)|df|_g^2), \tag{16}$$

where the curvatures and covariant derivatives on the right are those of g , and $\Delta f = \text{div}(\text{grad}f)$ is the Laplacian of f . If in addition $n \geq 3$, then

$$\widetilde{P} = P - \nabla^2 f + (df \otimes df) - \frac{1}{2}|df|_g^2, \tag{17}$$

$$\widetilde{W} = e^{2f}W. \tag{18}$$

In the pseudo-Riemannian case, the same formulas hold with each occurrence of $|df|_g^2$ replaced by $\langle df, df \rangle_g^2$.

3 The Problem

Consider constant Gaussian curvature so that the locally Euclidean metric of the manifold is scaled by a conformal factor. Scalar curvature results in a PDE in the form of Liouville's equation whose solution is the metric being sought for the manifold of interest.

Assume from the form of Liouville's equation for isothermal coordinates and perform a change of variables $\log f \Rightarrow u$ to (13) to obtain

$$\Delta_0 = -Ke^{2u} \quad (19)$$

The conformal scaling by curvature that is of interest is described by $ds^2 = e^{2u(x^1, \dots, x^n)} \sum (dx^i)^2$, with $u = u(x^1, \dots, x^n) = u(r)$ where $r = \sqrt{\sum x^i^2}$. The form of $u(x^1, \dots, x^n)$ is the proper exact solution to the PDE (19) but in earnest, $e^{2u(x^1, \dots, x^n)}$ will effectively end up being computed. Constant Gaussian curvature K with scalar Ricci curvature $R = n(n-1)K$ is sought. The choice of metric scales the flat locally Euclidean metric by e^{2u} .

The problem could be written less precisely instead as

$$\Delta_0 = -K\lambda(u), \quad (20)$$

in order to make calculations simpler. However, specifying in (20) that $\lambda(u) = e^{2u}$ will make the calculations more specific, and this will lend to directly connecting to Riemann's 1854 result. After all, the complex exponential function is analytic for every complex number that may be input. Namely, the power series expansion of the exponential function converges for any complex number, which here is chosen to be $2u$.

3.1 Formulae from Kreyszig [2]

Here it is denoted $\partial_i = \frac{\partial}{\partial x^i}$; $\partial_i u = u_i = \frac{\partial u}{\partial x^i}$; $u_{ij} = \partial_i \partial_j u$.

Let $\Delta u = \delta^{ij} \partial_i \partial_j$ denotes the Laplacian in flat coordinates.

Let $|\nabla u|^2 = \delta^{ij} u_i u_j$ denotes the Gradient.

$|\nabla u|^2 = \sum_m u^m u^m$.

Sometimes it will be written $r^2 = \sum x^2$.

The Kronecker delta is as usual:

$$\delta_\alpha^\beta = \begin{cases} 0 & (\alpha \neq \beta) \\ 1 & (\alpha = \beta) \end{cases} \quad (21)$$

$\mathbf{x}_\alpha \equiv \frac{\partial \mathbf{x}}{\partial \mathbf{u}^\alpha}$, $\mathbf{x}_{ab} \equiv \frac{\partial^2 \mathbf{u}}{\partial \mathbf{u}^\alpha \partial \mathbf{u}^\beta}$

$\mathbf{x}_\gamma \cdot \mathbf{x}^\kappa = \delta_\gamma^\kappa$ where $\mathbf{x}^\kappa = g^{\rho\kappa} \mathbf{x}_\rho$.

$g_{\alpha\gamma} g^{\gamma\beta} = \delta_\alpha^\beta$.

Christoffel symbols of the first kind:

$$\Gamma_{\alpha\beta\sigma} \equiv \mathbf{x}_{\alpha\beta} \cdot \mathbf{x}_\sigma = \frac{1}{2} \left(\frac{\partial g_{\beta\sigma}}{\partial u^\alpha} + \frac{\partial g_{\sigma\alpha}}{\partial u^\beta} - \frac{\partial g_{\alpha\beta}}{\partial u^\sigma} \right) \quad (22)$$

Transformation laws (i.e., "raising indices" and "lowering indices"):

$$\Gamma_{\alpha\beta}^\gamma = g^{\sigma\gamma} \Gamma_{\alpha\beta\sigma} \quad (23)$$

$$\Gamma_{\alpha\beta\gamma} = g_{\gamma\sigma} \Gamma_{\alpha\beta}^\sigma \quad (24)$$

The Mixed Riemann curvature tensor:

$$R_{\alpha\lambda}^\sigma = \frac{\partial \Gamma_{\alpha\beta}^\sigma}{\partial u^\lambda} - \frac{\partial \Gamma_{\alpha\lambda}^\sigma}{\partial u^\beta} + \Gamma_{\alpha\beta}^\kappa \Gamma_{\kappa\lambda}^\sigma - \Gamma_{\alpha\lambda}^\kappa \Gamma_{\kappa\beta}^\sigma \quad (25)$$

4 Solving the Problem

Scalar curvature results in a PDE whose solution is the metric being sought for the manifold of interest. Scalar curvature of a manifold comes from the Ricci tensor which comes from the Riemann curvature tensor which starts with Christoffel symbols.

From (22) \Rightarrow

$$\Gamma_{\alpha\beta\sigma} = \frac{1}{2} \left(\frac{\partial g_{\beta\sigma}}{\partial u^\alpha} + \frac{\partial g_{\sigma\alpha}}{\partial u^\beta} - \frac{\partial g_{\alpha\beta}}{\partial u^\sigma} \right) \quad (26)$$

$$(23) \Rightarrow \Gamma_{\alpha\beta}^\gamma = g^{\sigma\gamma} \Gamma_{\alpha\beta\sigma} = \frac{1}{2} g^{\sigma\gamma} \left(\frac{\partial g_{\beta\sigma}}{\partial u^\alpha} + \frac{\partial g_{\sigma\alpha}}{\partial u^\beta} - \frac{\partial g_{\alpha\beta}}{\partial u^\sigma} \right) \quad (27)$$

For the metric $g^{\sigma\gamma}$, the Christoffel symbols are

$$\Gamma_{\alpha\beta}^\gamma = \frac{1}{2} g^{\sigma\gamma} (\partial_\alpha g_{\beta\sigma} + \partial_\beta g_{\sigma\alpha} - \partial_\sigma g_{\alpha\beta}) \quad (28)$$

Switch from greek indices per Kreyszig's book to latin indices: $\alpha \Rightarrow i, \beta \Rightarrow k, \gamma \Rightarrow l, \sigma \Rightarrow m$.

For the metric g , the Christoffel symbols are

$$\Gamma_{ik}^l = \frac{1}{2} g^{ml} (\partial_i g_{km} + \partial_k g_{mi} - \partial_m g_{ik}) \quad (29)$$

Start with generic metric components:

$$g_{ik} = e^{2u} \delta_{ik} \quad (30)$$

Thus, the derivatives of (30) are:

$$\partial_i g_{mk} = 2e^{2u} \partial_i u \delta_{mk} = 2e^{2u} \delta_{mk} \partial_i u \quad (31)$$

$$\partial_k g_{mi} = 2e^{2u} \partial_k u \delta_{mi} = 2e^{2u} \delta_{mi} \partial_k u \quad (32)$$

$$\partial_m g_{ik} = 2e^{2u} \partial_m u \delta_{ik} = 2e^{2u} \delta_{ik} \partial_m u \quad (33)$$

Substituting these derivatives of the metric components into the Christoffel symbol expression (29):

$$(28) \Rightarrow \Gamma_{ik}^l = \frac{1}{2} e^{-2u} \delta^{lm} (2e^{2u} \delta_{mk} \partial_i u + 2e^{2u} \delta_{mi} \partial_k u + 2e^{2u} \delta_{ik} \partial_m u) \quad (34)$$

$$\Rightarrow \Gamma_{ik}^l = \delta^{lm} (\delta_{mk} \partial_i u + \delta_{mi} \partial_k u + \delta_{ik} \partial_m u) \quad (35)$$

Using the transformation law of Christoffel symbols $\delta^{lm} \delta_{mk} = \delta_k^l$ we get:

$$\delta^{lm} \delta_{mk} \partial_i u = \delta_k^l \partial_i u \quad (36)$$

$$\delta^{lm} \delta_{mi} \partial_k u = \delta_i^l \partial_k u \quad (37)$$

$$\delta^{lm} \delta_{ik} \partial_m u = \delta_m^l \partial_m u \quad (38)$$

Thus, (35) can be re-written as

$$(35) \Rightarrow \Gamma_{ik}^l = \delta_k^l \partial_i u + \delta_i^l \partial_k u - \delta_m^l \partial_m u \quad (39)$$

$$\Rightarrow \Gamma_{ik}^l = \delta_i^l \partial_k u + \delta_k^l \partial_i u - \delta_{ik} \delta^{lm} \partial_m u \quad (40)$$

$$\Rightarrow \Gamma_{ik}^l = \delta_i^l \partial_k u + \delta_k^l \partial_i u - \delta_{ik} \partial^l u \quad (41)$$

What has been calculated thus far will now be inserted into (25) to expand the Mixed Riemann curvature tensor. The form of the Riemann tensor re-written from greek indices as in (25) in latin indices will be computed thus

$$R_{ijk}^l = \underbrace{\partial_j \Gamma_{ik}^l}_{\textcircled{1}} - \underbrace{\partial_k \Gamma_{ij}^l}_{\textcircled{1}} + \underbrace{\Gamma_{ik}^m \Gamma_{mj}^l}_{\textcircled{2}} - \underbrace{\Gamma_{ij}^m \Gamma_{mk}^l}_{\textcircled{3}} \quad (42)$$

The necessary terms are computed using (41).

$$(41) : \Gamma_{ik}^l = \delta_i^l \partial_k u + \delta_k^l \partial_i u - \delta_{ik} \partial^l u$$

Derivatives ① & ②:

$$\textcircled{0} : \partial_j \Gamma_{ik}^l = \delta_i^l \partial_j \partial_k u + \delta_k^l \partial_j \partial_i u - \delta_{ik} \partial_j \partial^l u \quad (43)$$

$$\textcircled{1} : \partial_k \Gamma_{ij}^l = \delta_i^l \partial_k \partial_j u + \delta_j^l \partial_k \partial_i u - \delta_{ij} \partial_k \partial^l u \quad (44)$$

Connection terms ② & ③:

$$\textcircled{2} : \Gamma_{ik}^m \Gamma_{mj}^l = (\delta_i^m \partial_k u + \delta_k^m \partial_i u - \delta_{ik} \partial^m u)(\delta_m^l \partial_j u + \delta_j^l \partial_m u - \delta_{mj} \partial^l u) \quad (45)$$

$$\textcircled{3} : \Gamma_{ij}^m \Gamma_{mk}^l = (\delta_i^m \partial_j u + \delta_j^m \partial_i u - \delta_{ij} \partial^m u)(\delta_m^l \partial_k u + \delta_k^l \partial_m u - \delta_{mk} \partial^l u) \quad (46)$$

$$\text{Expand } \Gamma_{ik}^m \Gamma_{mj}^l = \underbrace{(\delta_i^m u_k)}_{\textcircled{0}} + \underbrace{\delta_k^m u_i}_{\textcircled{1}} - \underbrace{\delta_{ik} u_m}_{\textcircled{2}} \underbrace{(\delta_m^l u_j)}_{\textcircled{3}} + \underbrace{\delta_j^l u_m}_{\textcircled{4}} - \underbrace{\delta_{mj} u^l}_{\textcircled{5}}$$

$$\textcircled{0} \cdot \textcircled{3} : (\delta_i^m u_k)(\delta_m^l u_j) = \delta_i^l u_k u_j \quad (47)$$

$$\textcircled{0} \cdot \textcircled{4} : (\delta_i^m u_k)(\delta_j^l u_m) = \delta_j^l u_k u_i \quad (48)$$

$$\textcircled{0} \cdot \textcircled{5} : (\delta_i^m u_k)(-\delta_{mj} u^l) = -\delta_{ij}^l u_k u^l \quad (49)$$

$$\textcircled{1} \cdot \textcircled{3} : (\delta_k^m u_i)(\delta_m^l u_j) = \delta_k^l u_i u_j \quad (50)$$

$$\textcircled{1} \cdot \textcircled{4} : (\delta_k^m u_i)(\delta_j^l u_m) = \delta_j^l u_i u_k \quad (51)$$

$$\textcircled{1} \cdot \textcircled{5} : (\delta_k^m u_i)(-\delta_{mj} u^l) = -\delta_{kj}^l u_i u^l \quad (52)$$

$$\textcircled{2} \cdot \textcircled{3} : (-\delta_{ik} u^m)(\delta_m^l u_j) = -\delta_{ik} u^l u_j \quad (53)$$

$$\textcircled{2} \cdot \textcircled{4} : (-\delta_{ik} u^m)(\delta_j^l u_m) = -\delta_{ik} u^l u_j \quad (54)$$

$$\textcircled{2} \cdot \textcircled{5} : (-\delta_{ik} u^m)(-\delta_{mj} u^l) = -\delta_{ik} \delta_{mj} u^m u^l \quad (55)$$

Let $|\nabla u|^2 = \sum_m u^m u^m$. Sum (47) through (55) & simplify:

$$\Gamma_{ik}^m \Gamma_{mj}^l = \delta_i^l u_j u_k + \delta_j^l u_i u_k + \delta_k^l (u_j u_i + u_i u_j) - u^l (\delta_{ik} u_j + \delta_{jk} u_i + 2\delta_{ij} u_k) + \delta_{ij} \delta_{kl} |\nabla u|^2 \quad (56)$$

Re-labelling the indices of (56) to ascertain the result of the expansion computation but for $\Gamma_{ij}^m \Gamma_{mk}^l$ instead yields

$$\Gamma_{ij}^m \Gamma_{mk}^l = \delta_i^l u_k u_j + \delta_k^l u_i u_j + \delta_j^l (u_k u_i + u_i u_k) - u^l (\delta_{ij} u_k + \delta_{kj} u_i + 2\delta_{ik} u_j) + \delta_{ik} \delta_{jl} |\nabla u|^2 \quad (57)$$

Expand (42) from (44) - (43) + (56) - (57):

$$R_{ijk}^l = (\delta_k^l u_{ij} - \delta_j^l u_{ik} + \delta_{ij} u_{jl}) \quad (58)$$

$$+ (\delta_i^l u_k u_j + \delta_k^l u_i u_j + 2\delta_i^l u_i u_k - u^l (\delta_{ij} u_k + \delta_{kj} u_i + 2\delta_{ik} u_j) + \delta_{ik} \delta_{jl} |\nabla u|^2) \quad (59)$$

$$- (\delta_i^l u_j u_k + \delta_j^l u_i u_k + 2\delta_k^l u_i u_j - u^l (\delta_{ik} u_j + \delta_{jk} u_i + 2\delta_{ij} u_k) + \delta_{ij} \delta_{kl} |\nabla u|^2) \quad (60)$$

This expression for the Riemann tensor has three parts, namely the linear 2nd derivative terms (58), what will be referred to as the first quadratic terms (59), and finally what will be referred to as the second quadratic terms (60). The tensor contraction is $R_{ij} = R_{ikj}^k$ being summed over k .

Set $l = k$ and sum over k : $R_{ij} = \sum_k R_{ikj}^k$. Substitute $l = k$ into each term & sum. Since k is a dummy index, handle each group separately using $\delta_k^k = n$ (i.e., the trace identity in n dimensions) and properties of δ_{ij} .

Next the tensor contraction will be performed piece-wise, starting with the linear terms.

$$(58) : \sum_k \underbrace{(\delta_k^k u_{ij})}_{\textcircled{0}} - \underbrace{\delta_j^k u_{ik}}_{\textcircled{1}} + \underbrace{\delta_{ij} u_{kj}}_{\textcircled{2}} - \underbrace{\delta_{ik} u_{jk}}_{\textcircled{3}} \quad (61)$$

$$\textcircled{0} : \sum_k \delta_k^k u_{ij} = nu_{ij} \quad (62)$$

$$\textcircled{1} : - \sum_k \delta_j^k u_{ik} = -u_{ij} \quad (63)$$

$$\textcircled{2} : \sum_k \delta_{ij} u_{kj} = \delta_{ij} \sum_k u_{kj} = \delta_{ij} \Delta u \quad (64)$$

$$\textcircled{3} : - \sum_k \delta_{ik} u_{jk} = -u_{ji} \quad (65)$$

Summing (62) through (65) yields

$$(61) \Rightarrow \sum_k (\delta_k^k u_{ij} - \delta_j^k u_{ik} + \delta_{ij} u_{kj} - \delta_{ik} u_{jk}) = nu_{ij} - u_{ij} + \delta_{ij} \Delta u - u_{ji} \quad (66)$$

$$= (n-2)u_{ij} + \delta_{ij} \Delta u \quad (67)$$

Next the tensor contraction for the first quadratic terms is performed as follows:

$$\begin{aligned} (59) & : (\delta_i^l u_k u_j + \delta_k^l u_i u_j + 2\delta_i^l u_i u_k - u^l (\delta_{ij} u_k + \delta_{kj} u_i + 2\delta_{ik} u_j) + \delta_{ik} \delta_{jl} |\nabla u|^2) \\ \Rightarrow \sum_k & (\underbrace{\delta_i^k u_k u_j}_{\textcircled{0}} + \underbrace{\delta_k^k u_i u_j}_{\textcircled{1}} + \underbrace{2\delta_j^k u_i u_k}_{\textcircled{2}} - \underbrace{u_k (\delta_{ij} u_k + \delta_{kj} u_i + 2\delta_{ik} u_j)}_{\textcircled{3}} + \underbrace{\delta_{ik} \delta_{jk} |\nabla u|^2}_{\textcircled{6}}) \end{aligned} \quad (68)$$

$$\textcircled{0} : \sum_k \delta_i^k u_k u_j = u_i u_j \quad (69)$$

$$\textcircled{1} : \sum_k \delta_k^k u_i u_j = nu_i u_j \quad (70)$$

$$\textcircled{2} : \sum_k 2\delta_i^l u_i u_k = 2u_i u_j \quad (71)$$

$$\textcircled{3} : - \sum_k u^k \delta_{ij} u_k = -\delta_{ij} \sum_k u_k u_k = \delta_{ij} |\nabla u|^2 \quad (72)$$

$$\textcircled{4} : - \sum_k u_k \delta_{kj} u_i = -u_j u_i \quad (73)$$

$$\textcircled{5} : - \sum_k u^k 2\delta_{ik} u_j = -2u_i u_j \quad (74)$$

$$\textcircled{6} : - \sum_k \delta_{ik} \delta_{jk} |\nabla u|^2 = \delta_{ij} |\nabla u|^2 \quad (75)$$

Summing (69) through (75) yields

$$(59) \Rightarrow u_i u_j + nu_i u_j + 2u_i u_j - \delta_{ij} |\nabla u|^2 - u_j u_i - 2u_i u_j + \delta_{ij} |\nabla u|^2 \quad (76)$$

$$= (n+1)u_i u_j - u_i u_j = nu_i u_j \quad (77)$$

Similarly for the tensor contraction of the second quadratic terms

$$\begin{aligned} (60) & : -(\delta_i^l u_j u_k + \delta_j^l u_i u_k + 2\delta_k^l u_i u_j - u^l (\delta_{ik} u_j + \delta_{jk} u_i + 2\delta_{ij} u_k) + \delta_{ij} \delta_{kl} |\nabla u|^2) \\ \Rightarrow - \sum_k & (\underbrace{\delta_i^k u_j u_k}_{\textcircled{0}} + \underbrace{\delta_j^k u_i u_k}_{\textcircled{1}} + \underbrace{2\delta_k^k u_i u_j}_{\textcircled{2}} - \underbrace{u^k (\delta_{ik} u_j + \delta_{jk} u_i + 2\delta_{ij} u_k)}_{\textcircled{3}} + \underbrace{\delta_{ij} \delta_{kl} |\nabla u|^2}_{\textcircled{6}}) \end{aligned} \quad (78)$$

$$\textcircled{0} : - \sum_k \delta_i^k u_j u_k = -u_j u_i \quad (79)$$

$$\textcircled{1} : - \sum_k \delta_j^k u_i u_k = -u_i u_j \quad (80)$$

$$\textcircled{2} : - \sum_k 2\delta_k^k u_i u_j = -2nu_i u_j \quad (81)$$

$$\textcircled{3} : + \sum_k u^k \delta_{ik} u_j = u_i u_j \quad (82)$$

$$\textcircled{4} : + \sum_k u^k \delta_{jk} u_i = u_j u_i \quad (83)$$

$$\textcircled{5} : + \sum_k 2\delta_{ij} u^k u_k = 2\delta_{ij} |\nabla u|^2 \quad (84)$$

$$\textcircled{6} : - \sum_k \delta_{ij} \delta_{kj} |\nabla u|^2 = -\delta_{ij} |\nabla u|^2 \quad (85)$$

Summing (79) through (85) yields

$$(60) \Rightarrow -u_j u_i - u_i u_j - 2nu_i u_j + u_i u_j + u_j u_i + 2\delta_{ij} |\nabla u|^2 - \delta_{ij} |\nabla u|^2 = -2nu_i u_j + \delta_{ij} |\nabla u|^2 \quad (86)$$

Thus, the total R_{ij} from (67) + (77) + (86) is

$$R_{ij} = (n-2)u_{ij} + \delta_{ij} \Delta u + nu_i u_j - (2nu_i u_j - \delta_{ij} |\nabla u|^2) \quad (87)$$

$$= (n-2)u_{ij} + \delta_{ij} (\Delta u + |\nabla u|^2) - nu_i u_j \quad (88)$$

$$= (n-2)(u_{ij} - u_i u_j) + \delta_{ij} (\Delta u - (n-2)u_i u_j + |\nabla u|^2 - |\nabla u|^2) \quad (89)$$

$$= (n-2)(u_{ij} - u_i u_j) + \delta_{ij} (\Delta u - (n-2)|\nabla u|^2) \quad (90)$$

The Ricci Tensor is the trace $R_{ij} = R_{ikj}^k$. Scalar curvature follows from now implementing the conformal metric Ricci tensor:

$$R = g^{ij} R_{ij} = e^{-2u} \delta^{ij} [(n-2)(u_{ij} - u_i u_j) + \delta_{ij} (\Delta u - (n-2)|\nabla u|^2)] \quad (91)$$

$$= e^{-2u} [(n-2)(\Delta u - |\nabla u|^2) + n(\Delta u - (n-2)|\nabla u|^2)] \quad (92)$$

$$= e^{-2u} [-(n-1)(n-2)|\nabla u|^2 - 2(n-1)\Delta u] \quad (93)$$

For S^n , $R = n(n-1)K$ so the PDE may be expressed as:

$$-(n-1)(n-2)|\nabla u|^2 - 2(n-1)\Delta u = n(n-1)K e^{2u} \quad (94)$$

Compare (16) from earlier to (94) above to verify progress:

$$\tilde{S} = e^{-2f} (S - 2(n-1)\Delta f - (n-1)(n-2)|df|_g^2)$$

Divide (94) by $-2(n-1)$ to get:

$$\Delta u + \frac{n-2}{2} |\nabla u|^2 = -\frac{n}{2} K e^{2u} \quad (95)$$

The radial symmetry of S^n simplifies $|\nabla u|^2 = (u')^2$ and $\Delta u = u'' + \frac{n-1}{r} u'$. Thus (94) may now be written as

$$-(n-1)(n-2)(u')^2 - 2(n-1)(u'' + \frac{n-1}{r} u') = n(n-1)K e^{2u} \quad (96)$$

Next, a general stereographic projection form solution $e^{2u} = \frac{a^2}{(b+\alpha r^2)^2}$ is inserted into (96):

$$e^{2u} = \frac{a^2}{(b + \alpha r^2)^2} \Rightarrow e^u = \frac{a}{b + \alpha r^2} \quad (97)$$

$$\Rightarrow u = \ln a - \ln(b + \alpha r^2),$$

$$\Rightarrow u' = \frac{2\alpha r}{b + \alpha r^2},$$

$$\Rightarrow u'' = -\frac{2\alpha(b - \alpha r^2)}{(b + \alpha r^2)^2},$$

$$\Rightarrow |\nabla u|^2 = \frac{4\alpha^2 r^2}{(b + \alpha r^2)^2},$$

$$\Rightarrow \Delta u = -\frac{2\alpha(b - \alpha r^2)}{(b + \alpha r^2)^2} - \frac{2\alpha(n-1)}{b + \alpha r^2} = -\frac{2\alpha n}{(b + \alpha r^2)^2} + \frac{2\alpha^2 r^2(n-2)}{(b + \alpha r^2)^2} \quad (98)$$

Thus for the left hand side LHS of (96):

$$\frac{-4\alpha^2 r^2(n-1)(n-2) + 4\alpha n(n-1) - 4\alpha^2 r^2(n-1)(n-2)}{(b + \alpha r^2)^2} \Rightarrow \frac{4\alpha n(n-1)}{(b + \alpha r^2)^2} \quad (99)$$

For the right hand side RHS of (96):

$$n(n-1)\alpha \cdot \frac{a^2}{(b + \alpha r^2)^2} \quad (100)$$

Setting (99) = (100):

$$\alpha n(n-1) = n(n-1)\alpha a^2 \Rightarrow a^2 = 4 \quad (101)$$

Now set $b = 4$, $\alpha = 4K$ and factor:

$$e^{2u} = \frac{4}{(4 + 4Kr^2)^2} = \frac{1}{(1 + Kr^2)^2} \Rightarrow e^u = \frac{1}{1 + Kr^2} = \frac{1}{1 + \frac{\alpha}{4}r^2} \quad (102)$$

Thus, the conformal factor adjustment to the flat metric by the constant curvature

$$ds^2 = e^{2u} \sum dx^2 \Rightarrow ds = \frac{\sqrt{\sum (dx^i)^2}}{1 + \frac{\alpha}{4} \sum (x^i)^2} \quad (103)$$

5 Historical Parallel: Riemann's Über die Hypothesen

In "Part II, §4: Flat manifoldness may be treated as a special case of manifoldness with constant curvature" of his habilitation thesis, Riemann measures curvature via sectional planes. For constant curvature K , the result he presents in §4 is

$$ds^2 = \frac{\sum (dx^i)^2}{(1 + \frac{\alpha}{4} \sum (x^i)^2)^2}, \quad (104)$$

where $\alpha = 4K$.

6 Appendix: Stereographic Projection of S^n

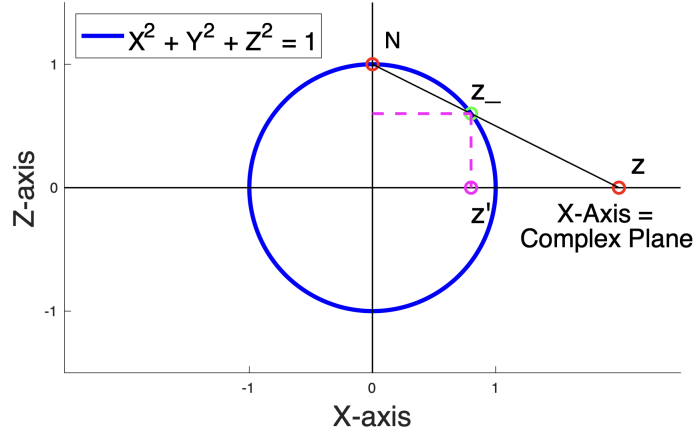


Figure 1: Midsagittal Section of the Stereographic Projection of z

Exercise. See figure 1. Let the projected point z be at $(x, y, 0)$, the projection point z_- is at (X, Y, Z) on S^3 ($\Rightarrow X^2 + Y^2 + Z^2 = 1$), and the image of the z' is at $X + iY$ in the complex-plane that the point z is “projecting” from.

Figure 1 shows that z and z_- are co-planar. From the similarity of the right triangles with hypotenuses Nz_- and Nz , the ratios of their legs parallel to the z -axis is given by

$$\frac{|z|}{|z'|} = \frac{1}{1 - Z}, \quad (105)$$

(105) can be interpreted as a scaling factor upon z' to obtain z , or conversely to scale $X + iY$ by a factor to obtain $x + iy$. Thus (105) \Rightarrow

$$\frac{|z|}{|z'|} z' = z \Leftrightarrow x + iy = \frac{X + iY}{1 - Z}. \quad (106)$$

Also, (105) \Rightarrow

$$\frac{|z|^2}{|z'|^2} = \frac{1}{(1 - Z)^2} \Rightarrow |z|^2 = \frac{|z'|^2}{(1 - Z)^2} \quad (107)$$

Notice that

$$X^2 + Y^2 + Z^2 = 1 \Rightarrow X^2 + Y^2 = 1 - Z^2 = (1 + Z)(1 - Z) \quad (108)$$

$$\Rightarrow \frac{X^2 + Y^2}{1 - Z} = 1 + Z \quad (109)$$

$$\Rightarrow \frac{X^2 + Y^2}{(1 - Z)^2} = \frac{1 + Z}{1 - Z} \quad (110)$$

Since $|z'|^2 = X^2 + Y^2$, (107) & (110) \Rightarrow

$$\begin{aligned} |z|^2 &= \frac{|z'|^2}{(1 - Z)^2} = \frac{X^2 + Y^2}{(1 - Z)^2} = \frac{1 + Z}{1 - Z} \\ \Rightarrow |z|^2 &= \frac{1 + Z}{1 - Z} \end{aligned} \quad (111)$$

Therefore, (111) \Rightarrow

$$\begin{aligned}
1 + |z|^2 &= \frac{1 - Z}{1 - Z} + \frac{1 + Z}{1 - Z} = \frac{2}{1 - Z} \\
\Rightarrow \frac{1}{1 + |z|^2} &= \frac{1 - Z}{2} = \frac{|z'|}{2|z|} \\
\Rightarrow \frac{2|z|}{1 + |z|^2} &= |z'| = \sqrt{X^2 + Y^2} \\
\Rightarrow \frac{2(x + iy)}{1 + x^2 + y^2} &= X + iY \tag{112}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \frac{2x}{1 + x^2 + y^2} + i \frac{2y}{1 + x^2 + y^2} &= X + iY \\
\Rightarrow X = \frac{2x}{1 + x^2 + y^2}, \quad Y = \frac{2y}{1 + x^2 + y^2} &\tag{113}
\end{aligned}$$

Notice that (112) resembles the form

$$\propto \frac{1}{1 + \sum x^2} \tag{114}$$

Also, (111) \Rightarrow

$$\begin{aligned}
|z|^2(1 - Z) &= 1 + Z \\
\Rightarrow |z|^2 - 1 &= (|z|^2 + 1)Z \\
\Rightarrow \frac{|z|^2 - 1}{|z|^2 + 1} &= Z \\
\Rightarrow \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} &= Z \tag{115}
\end{aligned}$$

Thus the stereographic formulae from (113) and (115) which determine the coordinates of the projection point (X, Y, Z) onto the unit sphere from the projected point at $(x, y, 0) \in \mathbb{C} \times \mathbb{R}$ are

$$\left(\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} \right) = (X, Y, Z) \tag{116}$$

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