MIT OCW 16.940J Computational Geometry Problem Set 1 Solutions

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1. **Show that the curvature of a planar curve is independent of the parametrization. Namely, if**

$$\mathbf{r}(t) = [x(t), y(t)] \tag{1}$$

is the curve then a change of variables

$$t = w(u) \text{ with } w'(u) \neq 0 \tag{2}$$

does not affect the curvature.

Curvature: $\kappa(t) = \dfrac{\sqrt{(\mathbf{r}' \cdot \mathbf{r}')(\mathbf{r}'' \cdot \mathbf{r}'') - (\mathbf{r}' \cdot \mathbf{r}'')^2}}{(\mathbf{r}' \cdot \mathbf{r}')^{3/2}}$. Note: planar curves have zero torsion, i.e., in this problem $\tau = 0$.

First, re-write the general non-arc-length parametrized formulation for curvature in two dimensions in terms of the components $x,\ y$ of $\mathbf{r}(t)$ as given.

$$\begin{aligned}
\kappa(t) &= \frac{\sqrt{((x', y') \cdot (x', y'))((x'', y'') \cdot (x'', y'')) - ((x', y') \cdot (x'', y''))^2}}{((x', y') \cdot (x'', y''))^{3/2}} \\
&= \frac{\sqrt{(x'^2) + y'^2)(x''^2) + y''^2) - (x'x'' + y'y'')^2}}{(x'^2 + y'^2)^{3/2}} \\
&= \frac{\sqrt{(x'^2x''^2 + x'^2y''^2 + y'^2x''^2 + y'^2y''^2) - ((x'x'')^2 + 2(x'x''y'y'') + (y'y'')^2)}}{(x'^2 + y'^2)^{3/2}} \\
&= \frac{\sqrt{(x'y'')^2 - 2(x'y''y'x'') + (y'x'')^2}}{(x'^2 + y'^2)^{3/2}} \\
\kappa(t) &= \frac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{3/2}} \qquad (\star)
\end{aligned}$$

Denote the change of variables $\mathbf{r}(t) \mapsto \mathbf{r}(w(u)) \equiv \mathbf{r}(u)$. Then next, compute $\mathbf{r}'(u)$ and $\mathbf{r}''(u)$ by the chain rule, use the results to recompute ($\star$), and finally compare the resulting curvature expression with ($\star$).

$$\begin{aligned}
\mathbf{r}'(u) &= \frac{d}{du}[x(w(u), y(w(u)] = [w'x'(u), w'x'(u)] = w'\mathbf{r}'(u) = w'[x'(u), y'(u)] \\
\mathbf{r}''(u) &= \frac{d}{du}[w'\mathbf{r}'(u)] = w''\mathbf{r}'(u) + (w')^2\mathbf{r}''(u) = w''[x'(u), y'(u)] + (w')^2[x''(u), y''(u)] \\
&= [w''x'(u) + (w')^2x''(u), w''y'(u) + (w')^2y''(u)]
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \kappa(u) &= \frac{|w'x'(u)(w''y'(u) + (w')^2y''(u)) - w'y'(u)(w''x'(u) + (w')^2x''(u))|}{((w'x'(u))^2 + (w'y'(u))^2)^{3/2}} \\
&= \frac{|w'[x'y'w'' + x'y''(w')^2 - x'y'w'' - y'x''(w')^2]|}{((w')^2(x'^2 + y'^2)^{3/2})} = \frac{|(w')^3||x'y'' - y'x''|}{|(w')^3|(x'^2 + y'^2)^{3/2}}
\end{aligned}$$

$$\Rightarrow \kappa(u) = \frac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{3/2}} = \kappa(t)$$

2. **Let a** [vector-parametrized] **curve $\mathbf{X}$ be defined by**

$$\mathbf{X}(t) = a \int \mathbf{g}(t) \times \mathbf{g}'(t)\ \mathrm{d}t,\ \ a = const. \neq 0, \tag{3}$$

where $\mathbf{g}(t)$ is a vector function satisfying $|\mathbf{g}(t)| = 1$ and $[\mathbf{g}\ \mathbf{g}'\ \mathbf{g}''] \neq 0$. Show that the curvature and the torsion of the curve are $\kappa \neq 0$ and $\tau = 1/a$, respectively.

Curvature: $\kappa(t) = \dfrac{\sqrt{(\mathbf{X}' \cdot \mathbf{X}')(\mathbf{X}'' \cdot \mathbf{X}'') - (\mathbf{X}' \cdot \mathbf{X}'')^2}}{(\mathbf{X}' \cdot \mathbf{X}')^{3/2}}$. Torsion: $\tau(t) = \dfrac{|\mathbf{X}'\mathbf{X}''\mathbf{X}'''|}{(\mathbf{X}' \times \mathbf{X}'') \cdot (\mathbf{X}' \times \mathbf{X}'')}$.
Compute the terms of the definitions above and then insert the results directly into the same definitions.

$$\mathbf{X}'(t) = a(\mathbf{g}(t) \times \mathbf{g}'(t))$$

$$\text{Identity: } A \times A = 0 \Rightarrow \mathbf{X}''(t) = a[\mathbf{g}' \times \mathbf{g}' + \mathbf{g} \times \mathbf{g}''] = a(\mathbf{g} \times \mathbf{g}'')$$

$$\text{Identity: } A \times B = -(B \times A) \Rightarrow \mathbf{X}'''(t) = a[\mathbf{g}'' \times \mathbf{g}' + \mathbf{g}' \times \mathbf{g}'' + \mathbf{g}' \times \mathbf{g}'' + \mathbf{g} \times \mathbf{g}'''] = a[\mathbf{g}' \times \mathbf{g}'' + \mathbf{g} \times \mathbf{g}'''].$$

Next, compute the dot products $\mathbf{X}' \cdot \mathbf{X}'$, $\mathbf{X}'' \cdot \mathbf{X}''$, and $\mathbf{X}' \cdot \mathbf{X}''$,

$$\begin{aligned}
\text{Identity: } (A \times B) \cdot C = C \cdot (A \times B) = |ABC| \Rightarrow \mathbf{X}' \cdot \mathbf{X}' &= a^2(\mathbf{g} \times \mathbf{g}') \cdot (\mathbf{g} \times \mathbf{g}') = a^2|\mathbf{g} \times \mathbf{g}'|^2 \\
\mathbf{X}'' \cdot \mathbf{X}'' &= a^2[\mathbf{g}' \times \mathbf{g}' + \mathbf{g} \times \mathbf{g}''] \cdot [\mathbf{g}' \times \mathbf{g}' + \mathbf{g} \times \mathbf{g}''] \\
&= a^2[\mathbf{g} \times \mathbf{g}''] \cdot [\mathbf{g} \times \mathbf{g}''] = a^2|\mathbf{g} \times \mathbf{g}''|^2 \\
\mathbf{X}' \cdot \mathbf{X}'' &= a^2(\mathbf{g} \times \mathbf{g}') \cdot [\mathbf{g}' \times \mathbf{g}' + \mathbf{g} \times \mathbf{g}''] \\
&= a^2(\mathbf{g} \times \mathbf{g}') \cdot (\mathbf{g} \times \mathbf{g}'').
\end{aligned}$$

Next, compute the cross-product of $\mathbf{X}'$ and $\mathbf{X}''$,

$$\begin{aligned}
\mathbf{X}' \times \mathbf{X}'' &= a(\mathbf{g} \times \mathbf{g}') \times a(\mathbf{g} \times \mathbf{g}'') \\
\mathbf{X}' \times \mathbf{X}'' &= a^2(\mathbf{g} \times \mathbf{g}') \times (\mathbf{g} \times \mathbf{g}'') \\
\text{Identity: } (A) \times (B \times C) = (A \cdot C)B - (A \cdot B)C \Rightarrow \mathbf{X}' \times \mathbf{X}'' &= a^2[((\mathbf{g} \times \mathbf{g}') \cdot \mathbf{g}'')\mathbf{g} - (\mathbf{g} \times \mathbf{g}') \cdot \mathbf{g})\mathbf{g}''] \\
\mathbf{g} \text{ is orthogonal to } \mathbf{g} \times \mathbf{g}' \Rightarrow (\mathbf{g} \times \mathbf{g}') \cdot \mathbf{g})\mathbf{g}'' = 0 \Rightarrow \mathbf{X}' \times \mathbf{X}'' &= a^2((\mathbf{g} \times \mathbf{g}') \cdot \mathbf{g}'')\mathbf{g} \\
\text{Identity: } (A \times B) \cdot C = C \cdot (A \times B) = |ABC| \Rightarrow \mathbf{X}' \times \mathbf{X}'' &= a^2|\mathbf{g}\ \mathbf{g}'\ \mathbf{g}''|\mathbf{g}.
\end{aligned}$$

Next, simplify the numerator of curvature κ and compute

$$\text{Identity: } |\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \Rightarrow \sqrt{(\mathbf{X}' \cdot \mathbf{X}')(\mathbf{X}'' \cdot \mathbf{X}'') - (\mathbf{X}' \cdot \mathbf{X}'')^2} = \sqrt{|\mathbf{X}' \times \mathbf{X}''|^2} = |\mathbf{X}' \times \mathbf{X}''|$$

$$\begin{aligned}
\mathbf{X}' \times \mathbf{X}'' = a^2|\mathbf{g}\ \mathbf{g}'\ \mathbf{g}''|\mathbf{g} &\Rightarrow |\mathbf{X}' \times \mathbf{X}''| = \sqrt{a^4|\mathbf{g}\ \mathbf{g}'\ \mathbf{g}''|^2|\mathbf{g}|} \\
|\mathbf{g}| = 1 &\Rightarrow |\mathbf{X}' \times \mathbf{X}''| = a^2|\mathbf{g}\ \mathbf{g}'\ \mathbf{g}''|.
\end{aligned}$$

It is given that a is a non-zero constant and that $[\mathbf{g}\ \mathbf{g}'\ \mathbf{g}''] \neq 0$, therefore $\kappa \neq 0$ if the denominator of curvature κ is also non-zero. Thus, simplify the denominator of curvature κ and compute

$$(\mathbf{X}' \cdot \mathbf{X}')^{3/2} = (|\mathbf{X}'|^2)^{3/2} = |\mathbf{X}'|^3.$$

Notice that $|\mathbf{X}'| = a|\mathbf{g} \times \mathbf{g}'| = a\sqrt{|\mathbf{g}|^2|\mathbf{g}'|^2 - (\mathbf{g} \cdot \mathbf{g}')^2}$. Since it is given that $|\mathbf{g}| = 1$ and $\mathbf{g}' \neq 0$ necessarily otherwise the curve is constant, $|\mathbf{X}'|$, the denominator of curvature κ and the entire expression for curvature κ itself are non-zero. Next, compute the denominator of torsion τ,

$$\begin{aligned}
\mathbf{X}' \times \mathbf{X}'' = a^2|\mathbf{g}\ \mathbf{g}'\ \mathbf{g}''|\mathbf{g} &\Rightarrow (\mathbf{X}' \times \mathbf{X}'') \cdot (\mathbf{X}' \times \mathbf{X}'') = a^4|\mathbf{g}\ \mathbf{g}'\ \mathbf{g}''|^2|\mathbf{g}|^2 \\
\text{From the given } |\mathbf{g}| = 1 &\Rightarrow (\mathbf{X}' \times \mathbf{X}'') \cdot (\mathbf{X}' \times \mathbf{X}'') = a^4|\mathbf{g}\ \mathbf{g}'\ \mathbf{g}''|^2.
\end{aligned}$$

Next, compute the numerator of torsion τ,

$$\begin{aligned}
\text{Identity: } (A \times B) \cdot C = C \cdot (A \times B) = |ABC| \Rightarrow |\mathbf{X}'\ \mathbf{X}''\ \mathbf{X}'''| &= (\mathbf{X}' \times \mathbf{X}'') \cdot \mathbf{X}''' \\
|\mathbf{X}'\ \mathbf{X}''\ \mathbf{X}'''| &= a^3|\mathbf{g}\ \mathbf{g}'\ \mathbf{g}''|[\mathbf{g} \cdot (\mathbf{g}' \times \mathbf{g}'') + \mathbf{g} \cdot (\mathbf{g} \times \mathbf{g}''')]
\end{aligned}$$

$\mathbf{g}$ is orthogonal to $\mathbf{g} \times \mathbf{g}'''$, and $\mathbf{g} \cdot (\mathbf{g}' \times \mathbf{g}'') = |\mathbf{g}\ \mathbf{g}'\ \mathbf{g}''| \Rightarrow |\mathbf{X}'\ \mathbf{X}''\ \mathbf{X}'''| = a^3|\mathbf{g}\ \mathbf{g}'\ \mathbf{g}''|^2$.

Therefore, $\tau = \dfrac{a^3|\mathbf{g}\mathbf{g}'\mathbf{g}''|^2}{a^4|\mathbf{g}\mathbf{g}'\mathbf{g}''|^2} = 1/a.$

3. **Find the parametric equation of a curve whose curvature κ and torsion τ are respectively**

$$\kappa = \frac{a}{a^2+b^2}, \quad \tau = \frac{b}{a^2+b^2}, \tag{4}$$

where $a > 0$ and b are constants.

First, let $c = a^2 + b^2$ and notice that

$$\kappa^2 + \tau^2 = \frac{a^2}{(a^2+b^2)^2} + \frac{b^2}{(a^2+b^2)^2} = \frac{c}{c^2} = \frac{1}{c}.$$

Use the given curvature κ and torsion τ to formulate the Frenet-Serret (differential) equations and solve for the parametric curve $\mathbf{r}(s)$ where s denotes arc-length. The Frenet-Serret (hereafter "FS") equations form a system of linear ordinary differential equations from the first derivatives of $\mathbf{t} = \mathbf{v}_1$, the unit tangent vector, $\mathbf{p} = \mathbf{v}_2$, the unit principal normal vector, and $\mathbf{b} = \mathbf{v}_3$, the unit binormal vector. As unit vectors, $|\mathbf{v}_j| = 1$ for $j = 1, 2, 3$. The FS equations may be written as $\mathbf{v}'_j = \sum_{k=1}^{3} c_{jk}\mathbf{v}_k$ for $j = 1, 2, 3$ such that the c_{jk} form the skew-symmetric matrix C, i.e.,

$$C = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}.$$

Note that a prime (i.e., $\mathbf{v}'_j$) here denotes a derivative with respect to the arc length s of the curve $\mathbf{r}(s)$ implied by the FS equations. In total the FS system of differential equations are written $\mathbf{F}' = C\mathbf{F}$ where $\mathbf{F} = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]^\intercal$. Written in this form, $\mathbf{t} = \mathbf{v}_1 = \dfrac{d\mathbf{r}}{ds}$.

A general exponential solution satisfying the FS equations is sought in the form of $\mathbf{F}(s) = \mathbf{F}(0)e^{sC}$ where $\mathbf{F}(0) = [\mathbf{v}_1(0)\ \mathbf{v}_2(0)\ \mathbf{v}_3(0)]^\intercal$ is the initial FS frame a.k.a. "tpb frame" at arc-length $s = 0$. In the following, denote $\omega = \dfrac{1}{\sqrt{c}} = \dfrac{1}{\sqrt{a^2+b^2}}$. Begin by calculating the first, second, and third derivatives of $\mathbf{v}_1$.

$$\begin{aligned}
\mathbf{v}'_1 &= \kappa\mathbf{v}_2 \\
\Rightarrow \mathbf{v}''_1 &= \kappa\mathbf{v}'_2 = \kappa(-\kappa\mathbf{v}_1 + \tau\mathbf{v}_3) = -\kappa^2\mathbf{v}_1 + \kappa\tau\mathbf{v}_3 \\
\Rightarrow \mathbf{v}'''_1 &= -\kappa^2\mathbf{v}'_1 + \kappa\tau\mathbf{v}'_3 \\
&= -\kappa^2(\kappa\mathbf{v}_2) + \kappa\tau(-\tau\mathbf{v}_2) \\
&= -\kappa^3\mathbf{v}_2 - \kappa\tau^2\mathbf{v}_2 \\
&= -\kappa(\kappa^2+\tau^2)\mathbf{v}_2 \\
\kappa^2 + \tau^2 = \frac{1}{c} = \frac{1}{\omega^2} \Rightarrow \mathbf{v}'''_1 &= \frac{-\kappa}{\omega^2}\mathbf{v}_2 \\
\mathbf{v}'_1/\kappa = \mathbf{v}_2 \Rightarrow \mathbf{v}'''_1 &= \frac{-\kappa}{\omega^2}\frac{1}{\kappa}\mathbf{v}'_1 = \frac{-1}{\omega^2}\mathbf{v}'_1 \\
\Rightarrow \mathbf{v}'''_1 + \frac{1}{\omega^2}\mathbf{v}'_1 &= 0
\end{aligned}$$

Therefore, the characteristic equation of the FS system of differential equations for the given curvature and torsion with roots q satisfying

$$q^3 + \frac{1}{\omega^2}q = q(q^2 + \frac{1}{\omega^2}) = 0.$$

Thus the roots of this characteristic equation are

$$q = 0,\ \pm i\omega$$

Note that the root $q = 0$ gives a constant vector along the binormal $\mathbf{b} \equiv \mathbf{v}_3$.

The general exponential solution for $\mathbf{v}_1$ is such that $|\mathbf{v}_1| = 1$ and $|\mathbf{v}_1'| = \kappa = \dfrac{a}{c}$. Next, the constant from $|\mathbf{v}_1'| = \kappa * \text{constant}$ is considered.

$$\mathbf{v}_1(s) = (k\cos(\omega s),\ k\sin(\omega s),\ h)$$
$$|\mathbf{v}_1| = 1 \Rightarrow |\mathbf{v}_1|^2 = k^2\cos^2(\omega s) + k^2\sin^2(\omega s) + h^2 = k^2 + h^2 = 1$$

Due to $|\mathbf{v}_1'| = \kappa \cdot \text{constant}$,

$$\begin{aligned} |\mathbf{v}_1'| &= k\omega = \kappa = \frac{a}{c} \\ k &= \frac{a}{c\omega} = \frac{a}{c \cdot \frac{1}{\sqrt{c}}} = \frac{a}{\sqrt{c}} \\ \Rightarrow k^2 &= \frac{a^2}{c} \\ k^2 + h^2 = 1 \Rightarrow h^2 &= 1 - \frac{a^2}{c} = \frac{c - a^2}{c} = \frac{b^2}{c} \end{aligned}$$

Thus, $\mathbf{v}_1(s) = \left(\dfrac{a}{\sqrt{c}}\cos(\omega s), \dfrac{a}{\sqrt{c}}\sin(\omega s), \dfrac{b}{\sqrt{c}}\right)$.
Integrating the tangent vector with respect to arc length yields the parametrized curve of interest, thus,

$$\begin{aligned} \mathbf{r}(s) = \int \mathbf{v}_1(s)\ ds &= \int \left(\frac{a}{\sqrt{c}}\cos(\omega s),\ \frac{a}{\sqrt{c}}\sin(\omega s),\ \frac{b}{\sqrt{c}}\right) ds \\ &= \left(\frac{a}{\sqrt{c}\omega}\sin(\omega s),\ -\frac{a}{\sqrt{c}\omega}\cos(\omega s),\ \frac{b}{\sqrt{c}}s\right) + \mathbf{D} \\ &= \left(a\sin(\frac{s}{\sqrt{a^2+b^2}}),\ -a\cos(\frac{s}{\sqrt{a^2+b^2}}),\ \frac{bs}{\sqrt{a^2+b^2}}\right) + \mathbf{D} \end{aligned}$$

where $\mathbf{D}$ is the integration constant. Choose $\mathbf{D} = [0\ 0\ 0]^\intercal$ to fix the start of the parametric curve at the origin. Thus, $\mathbf{r}(s) = \left(a\sin\left(\dfrac{s}{\sqrt{a^2+b^2}}\right),\ -a\cos\left(\dfrac{s}{\sqrt{a^2+b^2}}\right),\ \dfrac{bs}{\sqrt{a^2+b^2}}\right)$.

4. **A curve $\mathbf{C}_1$ is called an *involute* of a given curve C if tangents of C are normal to $\mathbf{C}_1$. The curve C is called an *evolute* of $\mathbf{C}_1$. Show that the curvature κ_1 of $\mathbf{C}_1$ is given by**

$$\kappa_1^2 = \frac{\kappa^2+\tau^2}{\kappa^2(c-s)^2}, \tag{5}$$

where c is constant, s is the arc length of C measured from a fixed point on C, and κ and τ are the curvature and torsion of C.

The position vector $\mathbf{r}(s)$ of the involute $\mathbf{C}_1$ at a point on $\mathbf{C}$ is $\mathbf{r}_1(s)$, namely

$$\mathbf{r}_1(s) = \mathbf{r}(s) + (c-s)\mathbf{T}(s)$$

where $\mathbf{T}(S)$ denotes the tangent vector for the displacement implied by $(c-s)$. Note that $\mathbf{T}'(s) = \kappa \cdot \mathbf{P}(s)$ is the relation between this tangent vector and the principal normal vector $\mathbf{P}(s)$. Furthermore, the arc length s_1 of $\mathbf{C}_1$ is related to s by $\frac{\mathrm{d}s_1}{\mathrm{d}s} = |\mathbf{r}_1'(s)| = |c-s|\kappa$. First, differentiate $\mathbf{r}_1(s)$,

$$\begin{aligned}\mathbf{r}_1'(s) = \frac{\mathrm{d}}{\mathrm{d}s}[\mathbf{r}(s)+(c-s)\mathbf{T}(s)] &= \mathbf{r}'(s) + (c-s)\mathbf{T}'(s) - \mathbf{T}(s)\\ &= (c-s)\kappa\mathbf{P}(s)\end{aligned}$$

Similarly, since $\mathbf{P}(s)$ is a unit vector (i.e., $|\mathbf{P}(s)| = 1$) and $\kappa > 0$, the magnitude of $\mathbf{r}_1'(s)$ is thus

$$|\mathbf{r}_1'(s)| = |(c-s)\kappa\mathbf{P}(s)| = |c-s|\kappa$$

Following this, the the unit tangent $\mathbf{T}_1$ of $\mathbf{C}_1$ is

$$\mathbf{T}_1(s) = \frac{\mathbf{r}_1'(s)}{|\mathbf{r}_1'(s)|} = \frac{(c-s)\kappa\mathbf{P}(s)}{|c-s|\kappa} = \mathbf{P}(s)$$

Next, compute curvature $\kappa_1 = \left|\frac{\mathrm{d}\mathbf{T}_1}{\mathrm{d}s}\right|$. By the FS equations,

$$\mathbf{T}_1 = \mathbf{P} \Rightarrow \frac{\mathrm{d}\mathbf{T}_1}{\mathrm{d}s} = \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}s} = -\kappa\mathbf{T} + \tau\mathbf{B}$$

where $\mathbf{B}$ denotes the binormal vector. By the chain rule for derivatives,

$$\frac{\mathrm{d}\mathbf{T}_1}{\mathrm{d}s_1} = \frac{\mathrm{d}\mathbf{T}_1}{\mathrm{d}s}\cdot\frac{\mathrm{d}s}{\mathrm{d}s_1} = \frac{\mathbf{P}'}{\mathrm{d}s_1/\mathrm{d}s} = \frac{-\kappa\mathbf{T}+\tau\mathbf{B}}{|c-s|\kappa}$$

Without loss of generality, assume that $c > s$. Next, to compute curvature, calculate the magnitude of $\frac{\mathrm{d}\mathbf{T}_1}{\mathrm{d}s_1}$,

$$\kappa_1 = \left|\frac{-\kappa\mathbf{T}+\tau\mathbf{B}}{(c-s)\kappa}\right| = \frac{|-\kappa\mathbf{T}+\tau\mathbf{B}|}{(c-s)\kappa}$$

$\mathbf{T}$ and $\mathbf{B}$ are orthonormal i.e., $|\mathbf{T}| = |\mathbf{B}| = 1$ and $\mathbf{T}\cdot\mathbf{B} = 0$, thus

$$\begin{aligned}\Rightarrow |-\kappa\mathbf{T}+\tau\mathbf{B}| &= \sqrt{(-\kappa)^2+\tau^2} = \sqrt{\kappa^2+\tau^2}\\ \Rightarrow \kappa_1^2 &= \frac{\kappa^2+\tau^2}{\kappa^2(c-s)^2}.\end{aligned}$$

5. **Let E, F, G be the coefficients of the first fundamental form of a regular surface $\mathbf{R} = \mathbf{R}(u, v)$. Let $f(u, v) = c$ and $g(u, v) = d$ be two families of regular curves defined in the parameter space $u - v$ of the surface with images in 3D space obtained for various constants c and d. Prove that the 3D images of these two families of curves are orthogonal (i.e., whenever two curves of distinct families meet, their tangents are orthogonal) if and only if**

$$E f_v g_v - F(f_u g_v + f_v g_u) + G f_u g_u = 0 \tag{6}$$

where $E = \mathbf{R}_u \cdot \mathbf{R}_u, F = \mathbf{R}_u \cdot \mathbf{R}_v, G = \mathbf{R}_v \cdot \mathbf{R}_v$, and subscripts u, v denote partial derivatives.

∇f is normal to the curve in the $u - v$ plane and is thus perpendicular to $(-f_v, f_u)$ and $(-g_v, g_u)$. Start by parametrizing and differentiating using the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}t} f(u(t), v(t)) = f_u u' + f_v v' = 0.$$

Thus, $(u', v') \propto (-f_v, f_u)$, so for scalar k

$$\begin{aligned} \mathbf{T}_f = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{R}(u(t), v(t)) &= \mathbf{R}_u u' + \mathbf{R}_v v' \\ &= k(-f_v \mathbf{R}_u + f_u \mathbf{R}_v) \end{aligned}$$

and similarly $u' g_u + v' g_v = 0$, then for scalar m

$$\mathbf{T}_g = \mathbf{R}_u u' + \mathbf{R}_v v' = m(-g_v \mathbf{R}_u + g_u \mathbf{R}_v).$$

Orthogonality in 3D requires that $\mathbf{T}_f \cdot \mathbf{T}_g = 0$, thus

$$\mathbf{T}_f \cdot \mathbf{T}_g = km[(-f_v \mathbf{R}_u + f_u \mathbf{R}_v) \cdot (-g_v \mathbf{R}_u + g_u \mathbf{R}_v)]$$

The bracketed term in the above expression is equivalent to

$$f_v g_v(\mathbf{R}_u \cdot \mathbf{R}_u) - f_v g_u(\mathbf{R}_u \cdot \mathbf{R}_v) - f_u g_v(\mathbf{R}_u \cdot \mathbf{R}_v) + f_u g_u(\mathbf{R}_v \cdot \mathbf{R}_v)$$

Thus, since it is given that $E = \mathbf{R}_u \cdot \mathbf{R}_u, F = \mathbf{R}_u \cdot \mathbf{R}_v, G = \mathbf{R}_v \cdot \mathbf{R}_v$,

$$\mathbf{T}_f \cdot \mathbf{T}_g = km[E f_v g_v - F(f_v g_u + f_u g_v) + G f_u g_u].$$

Note that the curves are orthogonal if and only if the bracketed term in the above expression equals 0 since $km \neq 0$ assuming nonzero tangent vectors. ■

6. **Consider a torus parametrized as follows:**

$$\mathbf{r}(u,v) = [(R + a\cos u)\cos v, (R + a\cos u)\sin v, a\sin u] \tag{7}$$

where $0 \leq u \leq 2\pi,\ 0 \leq v \leq 2\pi$, and R and a are constants such that $R > a$. Derive formulae for the Gauss, mean, and principal curvatures. Sketch the torus and subdivide it into hyperbolic, parabolic, and elliptic regions. In a follow-up sketch, illustrate the lines of curvature of the torus. Explain the above subdivision and sketches. [1]

For a torus embedded in 3D space, say $x_1x_2x_3$ space, there will be $3 - 1 = 2$ principal directions and corresponding principal curvatures. Denote the components of the first fundamental form g_{jk} for $j,\ k = 1,\ 2$ respectively and the components of the second fundamental form b_{jk} for $j,\ k = 1,\ 2$ respectively. In this case, j or k being 1 corresponds to u and likewise being 2 corresponds to v.

First, compute the first and second derivatives of $\mathbf{r}$ needed for the first and second fundamental forms,

$$\begin{aligned}
\mathbf{r}_u &= [(-a\sin u)\cos v, (-a\sin u)\sin v, a\cos u], \quad \mathbf{r}_v = [-(R + a\cos u)\sin v, (R + a\cos u)\cos v, 0],\\
\mathbf{r}_{uu} &= [-a\cos u\cos v, -a\cos u\sin v, -a\sin u],\ \ \mathbf{r}_{uv} = [a\sin u\sin v, -a\sin u\cos v, 0],\\
\mathbf{r}_{vv} &= [-(R + a\cos u)\cos v, -(R + a\cos u)\sin v, 0]
\end{aligned}$$

Next, compute the components of the first fundamental form and the discriminant of the first fundamental form,

$$\begin{aligned}
g_{11} &= \mathbf{r}_u \cdot \mathbf{r}_u = a^2\sin^2 u(\sin^2 v + \cos^2 v) + a^2\cos^2 u = a^2\\
g_{12} &= \mathbf{r}_u \cdot \mathbf{r}_v = ((a\sin u)\cos v)((R + a\cos u)\sin v) - ((a\sin u)\sin v)((R + a\cos u)\cos v) + a\cos u \cdot 0 = 0\\
g_{22} &= \mathbf{r}_v \cdot \mathbf{r}_v = (-(R + a\cos u)\sin v)^2 + ((R + a\cos u)\cos v)^2 + 0^2 = (R + a\cos u)^2\\
g &= g_{11}g_{22} - (g_{12})^2 = a^2(R + a\cos u)^2.
\end{aligned}$$

Next, compute the normal vector for the torus starting with the appropriate cross product of first derivatives of $\mathbf{r}$,

$$\begin{aligned}
\mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} i & j & k\\ (-a\sin u)\cos v & (-a\sin u)\sin v & a\cos u\\ -(R + a\cos u)\sin v & (R + a\cos u)\cos v & 0 \end{vmatrix}\\
&= [-a\cos u\cos v(R + a\cos u), -a\cos u\sin v(R + a\cos u), -a\sin u(R + a\cos u)]\\
\mathbf{n} = \frac{\mathbf{x}_1 \times \mathbf{x}_2}{+\sqrt{g}} &= \frac{[-a\cos u\cos v(R + a\cos u), -a\cos u\sin v(R + a\cos u), -a\sin u(R + a\cos u)]}{a(R + a\cos u)}\\
&= [-\cos u\cos v, -\cos u\sin v, -\sin u].
\end{aligned}$$

Next, use the previous results to compute the components of the second fundamental form $b_{11} = \mathbf{r}_{uu} \cdot \mathbf{n}$, $b_{12} = \mathbf{r}_{uv} \cdot \mathbf{n}$, $b_{22} = \mathbf{r}_{vv} \cdot \mathbf{n}$, and the discriminant of the second fundamental form $b = b_{11}b_{22} - b_{12}^2$,

$$\begin{aligned}
b_{11} &= a\cos^2 u\cos^2 v + a\cos^2 u\sin^2 v + a\sin^2 u = a\\
b_{12} &= a\sin u\cos u\sin v\cos v - a\sin u\cos u\sin v\cos v = 0\\
b_{22} &= (R + a\cos u)\cos u(\sin^2 v + \cos^2 v) = (R + a\cos u)\cos u\\
b &= a(R + a\cos u)\cos u
\end{aligned}$$

Thus, for the given parametrization of the torus the principal, Gauss, and mean curvatures are

$$\begin{aligned}
\kappa_1 &= \frac{b_{11}}{g_{11}} = -\frac{1}{a}, \quad \kappa_2 = \frac{b_{22}}{g_{22}} = -\frac{\cos u}{R + a\cos u}\\
\Rightarrow K &= \frac{b}{g} = \kappa_1\kappa_2 = \frac{\cos u}{a(R + a\cos u)}, \quad H = \frac{1}{2}(\kappa_1 + \kappa_2) = -\frac{1}{2}\left(\frac{-\cos u}{R + a\cos u} - \frac{1}{a}\right).
\end{aligned}$$

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[1] Problem 17 in "Shape Interrogation for Computer Aided Design and Manufacturing" by N. M. Patrikalakis and T. Maekawa, 1st ed., Springer, 2002.

If $v = \pm\frac{\pi}{2}$ then $K = 0$ i.e., circles at the maximum and minimum height a on the torus, say S, measured from the torus center perpendicular to R consist of parabolic points only. Points on S whose distance from the center of S to its height extrema along the x_3-axis in the figure below is greater than $\sqrt{R^2 + a^2}$ are elliptic while those whose distance from the origin/center of S is smaller than $\sqrt{R^2 + a^2}$ are hyperbolic. The lines of curvature on a torus consist of meridians and parallels. The curves of constant Gaussian curvature are the meridians (i.e, the lines of curvature which as plane curves lie in planes parallel to the $x_1 - x_2$ plane). Note that a second circle consisting of only parabolic points on the underside of the torus at its minimum z-axis height in the left figure below is not visible.

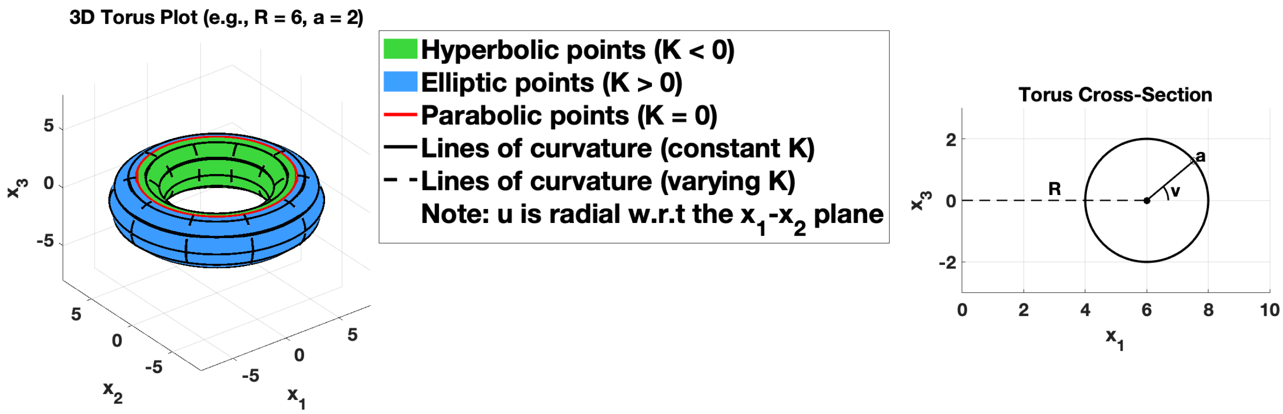
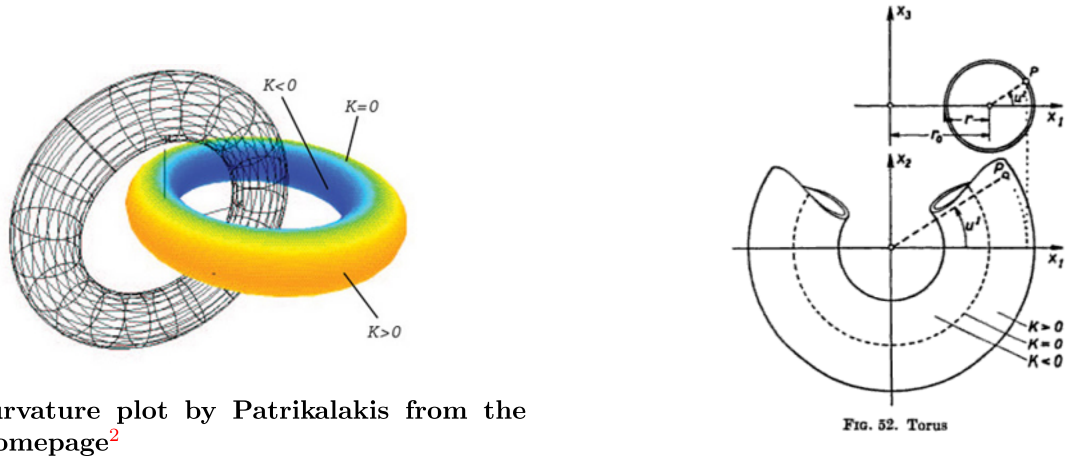


Figure 1: Problem 6 Torus Curvature Plots. Left: 3D Torus plot with $R = 6,\ a = 2$. Right: Torus Cross-Section.



Torus curvature plot by Patrikalakis from the course homepage[2]

Figure 52 from ***Differential Geometry*** **by Erwin Kreyszig** (§44. Torus)[3]

Figure 2: Reference figures for comparison

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[2] Image adapted from N. M. Patrikalakis, "Computational Geometry," MIT OpenCourseWare, Course 2.158J, Spring 2003, `https://ocw.mit.edu/courses/2-158j-computational-geometry-spring-2003/`.

[3] Image reproduced from Figure 52 in E. Kreyszig, *Differential Geometry*, Dover Publications, Inc., Mineola, NY, 1991 (unabridged republication of the 1963 edition), p. 136, Section 44.

7. **Show that the surface area on a Monge patch $\mathbf{X}(u,v) = u\mathbf{e}_1 + v\mathbf{e}_2 + f(u,v)\mathbf{e}_3$ is given by the integral**

$$A = \int\int_W \sqrt{1 + f_u^2 + f_v^2}\ \mathrm{d}u\ \mathrm{d}v, \tag{8}$$

where W is the parameter domain, and $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ are the unit coordinate vectors.

A "*Monge Patch*" refers to the parametrization of a surface by its height over a flat reference plane, and is sometimes also called a portion of a surface. Denote the components of the first fundamental form as $g_{11},\ g_{12} = g_{21}$, and g_{22}. Furthermore, denote the discriminant of the first fundamental form as $g = g_{11}g_{22} - (g_{12})^2$. Taking the definition of the area of a portion of a surface or Monge Patch,

$$A = \int\int_W \sqrt{g}\ \mathrm{d}u\ \mathrm{d}v$$

compute the are directly from the first fundamental form for $\mathbf{X}$. Begin by computing the derivatives with respect to u and v of $\mathbf{X}$ and then compute the first fundamental form,

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}u} &= \mathbf{e}_1 + f_u\mathbf{e}_3, \quad \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}v} = \mathbf{e}_2 + f_u\mathbf{e}_3 \\
g_{11} &= (\mathbf{e}_1 + f_u\mathbf{e}_3) \cdot (\mathbf{e}_1 + f_u\mathbf{e}_3) = 1 + f_u^2 \\
g_{22} &= (\mathbf{e}_2 + f_u\mathbf{e}_3) \cdot (\mathbf{e}_2 + f_u\mathbf{e}_3) = 1 + f_v^2 \\
g_{12} &= (\mathbf{e}_1 + f_u\mathbf{e}_3) \cdot (\mathbf{e}_2 + f_u\mathbf{e}_3) = f_u f_v \\
g &= (1 + f_u^2)(1 + f_v^2) - (f_u f_v)^2 = 1 + f_u^2 + f_v^2.
\end{aligned}$$

Re-inserting the resulting discriminant of the first fundamental form into $A = \int\int_W \sqrt{g}\ \mathrm{d}u\ \mathrm{d}v$ recapitulates (8).

8. **Show that the second fundamental form on a Monge patch $\mathbf{X}(u, v) = u\mathbf{e}_1 + v\mathbf{e}_2 + f(u, v)\mathbf{e}_3$ is**

$$II = (f_u^2 + f_v^2 + 1)^{-\frac{1}{2}} \left[f_{uu} du^2 + 2f_{uv} \mathrm{d}u^2 + 2f_{uv} \mathrm{d}u\mathrm{d}v + f_{vv} \mathrm{d}v^2 \right], \tag{9}$$

where $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ are the unit coordinate vectors.

For a given general vector parametrized curve $\mathbf{X}$, the second fundamental form is defined in Einstein notation (i.e., summation is implicit) by $b_{\alpha\beta}\mathrm{d}u^\alpha\ \mathrm{d}u^\beta$ where $\alpha, \beta = 1, 2$ respectively, namely

$$b_{\alpha\beta} = \mathbf{x}_{\alpha\beta} \cdot \mathbf{n} \text{ where } \mathbf{n} \text{ denotes the normal vector to } \mathbf{X},$$
$$II = b_{11}(\mathrm{d}u^1)^2 + 2b_{12}\mathrm{d}u^1\mathrm{d}u^2 + b_{22}(\mathrm{d}u^2)^2 = L(\mathrm{d}u^1)^2 + 2M\mathrm{d}u^1\mathrm{d}u^2 + N(\mathrm{d}u^2)^2.$$

Furthermore, the normal vector is generally defined by

$$\mathbf{n} = \frac{\mathbf{x}_1 \times \mathbf{x}_2}{|\mathbf{x}_1 \times \mathbf{x}_2|} = \frac{\mathbf{x}_1 \times \mathbf{x}_2}{+\sqrt{g}}$$

where g denotes the discriminant of the first fundamental form. First compute the first and second derivatives of the given $\mathbf{X}$, then directly compute the second fundamental form. Derivatives:

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}u} = \mathbf{e}_1 + f_u\mathbf{e}_3, \quad \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}v} = \mathbf{e}_2 + f_v\mathbf{e}_3,$$
$$\frac{\mathrm{d}^2\mathbf{X}}{\mathrm{d}u^2} = f_{uu}\mathbf{e}_3, \quad \frac{\mathrm{d}^2\mathbf{X}}{\mathrm{d}u\mathrm{d}v} = f_{uv}\mathbf{e}_3, \quad \frac{\mathrm{d}^2\mathbf{X}}{\mathrm{d}v^2} = f_{vv}\mathbf{e}_3.$$

Next, compute the cross product of the first derivatives of $\mathbf{X}$,

$$\begin{aligned}
\mathbf{X}_u \times \mathbf{X}_v &= (\mathbf{e}_1 + f_u\mathbf{e}_3) \times (\mathbf{e}_2 + f_v\mathbf{e}_3) \\
&= \mathbf{e}_1 \times \mathbf{e}_2 + \mathbf{e}_1 \times f_v\mathbf{e}_3 + f_u\mathbf{e}_3 \times \mathbf{e}_2 + f_u\mathbf{e}_3 \times f_v\mathbf{e}_3 \\
&= \mathbf{e}_3 - f_v\mathbf{e}_2 - f_u\mathbf{e}_1.
\end{aligned}$$

Next, compute the dot products of first derivatives of $\mathbf{X}$,

$$\begin{aligned}
\mathbf{X}_u' \cdot \mathbf{X}_u' &= (\mathbf{e}_1 + f_u\mathbf{e}_3) \cdot (\mathbf{e}_1 + f_u\mathbf{e}_3) \\
&= \mathbf{e}_1 \cdot \mathbf{e}_1 + \mathbf{e}_1 \cdot f_u\mathbf{e}_3 + f_u\mathbf{e}_3 \cdot \mathbf{e}_1 + f_u^2\mathbf{e}_3 \cdot \mathbf{e}_3 \\
&= 1 + f_u^2 \\
\mathbf{X}_u \cdot \mathbf{X}_v &= (\mathbf{e}_1 + f_u\mathbf{e}_3) \cdot (\mathbf{e}_2 + f_v\mathbf{e}_3) \\
&= \mathbf{e}_1 \cdot \mathbf{e}_2 + \mathbf{e}_1 \cdot f_v\mathbf{e}_3 + f_u\mathbf{e}_3 \cdot \mathbf{e}_2 + f_u f_v\mathbf{e}_3 \cdot \mathbf{e}_3 \\
&= f_u f_v \\
\mathbf{X}_v \cdot \mathbf{X}_v &= (\mathbf{e}_2 + f_v\mathbf{e}_3) \cdot (\mathbf{e}_2 + f_v\mathbf{e}_3) \\
&= \mathbf{e}_2 \cdot \mathbf{e}_2 + \mathbf{e}_2 \cdot f_v\mathbf{e}_3 + f_v\mathbf{e}_3 \cdot \mathbf{e}_2 + f_v^2\mathbf{e}_3 \cdot \mathbf{e}_3 \\
&= 1 + f_v^2.
\end{aligned}$$

Next, compute the discriminant of the first fundamental form,

$$\begin{aligned}
g &= g_{11}g_{22} - (g_{12})^2 \\
g &= (1 + f_u^2)(1 + f_v^2) - (f_u f_v)^2 \\
&= 1 + f_u^2 + f_v^2 + f_u^2 f_v^2 - (f_u^2 f_v^2) \\
&= f_u^2 + f_v^2 + 1 \\
\Rightarrow \sqrt{g} &= \sqrt{f_u^2 + f_v^2 + 1}.
\end{aligned}$$

Next, compute the normal vector,

$$\mathbf{n} = \frac{\mathbf{e}_3 - f_v\mathbf{e}_2 - f_u\mathbf{e}_1}{\sqrt{f_u^2 + f_v^2 + 1}}$$

Note that

$$\mathbf{e}_3 \cdot (\mathbf{e}_3 - f_v \mathbf{e}_2 - f_u \mathbf{e}_1) = 1 \tag{$\star\star$}$$

Finally, compute the second fundamental form from the definition and the results above, and compare with (9)

$$\begin{aligned}
f_{uu}\mathbf{e}_3 \cdot \mathbf{n} &= \frac{f_{uu}}{\sqrt{f_u^2 + f_v^2 + 1}} \cdot (\star\star) \\
f_{uv}\mathbf{e}_3 \cdot \mathbf{n} &= \frac{f_{uv}}{\sqrt{f_u^2 + f_v^2 + 1}} \cdot (\star\star) \\
f_{vv}\mathbf{e}_3 \cdot \mathbf{n} &= \frac{f_{vv}}{\sqrt{f_u^2 + f_v^2 + 1}} \cdot (\star\star) \\
\Rightarrow II &= (f_u^2 + f_v^2 + 1)^{-\frac{1}{2}} \left[f_{uu} du^2 + 2f_{uv}\mathrm{d}u^2 + 2f_{uv}\mathrm{d}u\mathrm{d}v + f_{vv}\mathrm{d}v^2 \right].
\end{aligned}$$

9. **Show that the principal curvatures of the surface** $f(x, y, z) = x\sin(z) - y\cos(z) = 0$ **are** $\pm(x^2+y^2+1)^{-1}$**.**

Introduce a change of variables to cylindrical coordinates given by $x = r\cos\theta$, $y = r\sin\theta$, $\theta = z - n\pi$,

$$f(x, y, z) \mapsto \mathbf{r}(r, z) = (r\cos(z - n\pi), r\sin(z - n\pi), z).$$

This parametrization satisfies the original $f(x, y, z)$. Without loss of generality, let $n = 0$, thus the change of variables for the given surface parametrization may be written

$$\mathbf{r}(r, z) = (r\cos z, r\sin z, z).$$

The coefficients of the first fundamental form are thus

$$E = \mathbf{r}_r \cdot \mathbf{r}_r,\ F = \mathbf{r}_r \cdot \mathbf{r}_z,\ G = \mathbf{r}_z \cdot \mathbf{r}_z.$$

Proceed to compute the first derivatives in the above expressions, the coefficients of the first fundamental form, and the subsequent first fundamental form:

$$\begin{aligned} \mathbf{r}_r = (\cos z, \sin z, 0),\ \mathbf{r}_z &= (-r\sin z, r\cos z, 1) \\ E &: \cos^2 z + \sin^2 z + 0 = 1 \\ F &: -r\cos z\sin z + r\sin z\cos z = 0 \\ G &: (-r\sin z)^2 + (r\cos z)^2 + 1^2 = r^2 + 1 \\ &\Rightarrow I = \mathrm{d}s^2 = \mathrm{d}r^2 + (r^2+1)\mathrm{d}z^2. \end{aligned}$$

Next, compute the normal vector:

$$\begin{aligned} \mathbf{r}_r \times \mathbf{r}_z &= \begin{vmatrix} i & j & k \\ \cos z & \sin z & 0 \\ -r\sin z & r\cos z & 1 \end{vmatrix} = (\sin z, -\cos z, r) \\ \Rightarrow |\mathbf{r}_r \times \mathbf{r}_z| &= \sqrt{\sin^2 z + \cos^2 z + r^2} = \sqrt{1+r^2} \\ \Rightarrow \mathbf{n} &= \frac{(\sin z, -\cos z, r)}{\sqrt{1+r^2}}. \end{aligned}$$

Next, compute the second derivatives of $\mathbf{r}$, the coefficients of the second fundamental form, and the subsequent second fundamental form

$$\begin{aligned} \mathbf{r}_{rr} &= (0, 0, 0) \\ \mathbf{r}_{rz} &= (-\sin z, \cos z, 0) \\ \mathbf{r}_{zz} &= (-r\cos z, -r\sin z, 0) \\ L : \mathbf{n} \cdot \mathbf{r}_{rr} &= \mathbf{n} \cdot (0, 0, 0) = 0 \\ M : \mathbf{n} \cdot \mathbf{r}_{rz} &= \frac{(\sin z)(-\sin z) + (-\cos z)(\cos z) + (r)(0)}{\sqrt{1+r^2}} \\ &= \frac{-\sin^2 z - \cos^2 z}{\sqrt{1+r^2}} = \frac{-1}{\sqrt{1+r^2}} \\ N : \mathbf{n} \cdot \mathbf{r}_{zz} &= \frac{(\sin z)(-r\cos z) + (-\cos z)(-r\sin z) + (r)(0)}{\sqrt{1+r^2}} \\ &= \frac{-r\sin z\cos z + r\cos z\sin z}{\sqrt{1+r^2}} = 0 \\ \Rightarrow II &= \frac{-1}{\sqrt{1+r^2}}\mathrm{d}r\ \mathrm{d}z. \end{aligned}$$

The principal curvatures can be calculated as the eigenvalues of the shape operator given by the Weingarten matrix $W = I^{-1}II$ where in this case I and II are the matrices of the 1^{st} and 2^{nd} fundamental forms, respectively.

$$I = \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & r^2+1 \end{bmatrix};\ I^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{r^2+1} \end{bmatrix}$$

$$II = \begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{\sqrt{1+r^2}} \\ \frac{-1}{\sqrt{1+r^2}} & 0 \end{bmatrix}$$

$$\Rightarrow W = I^{-1}II = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{r^2+1} \end{bmatrix} \begin{bmatrix} 0 & \frac{-1}{\sqrt{1+r^2}} \\ \frac{-1}{\sqrt{1+r^2}} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{\sqrt{1+r^2}} \\ \frac{-1}{(r^2+1)\sqrt{1+r^2}} & 0 \end{bmatrix}.$$

Next, compute the eigenvalues of W,

$$\det(W - \lambda I) = \det\begin{pmatrix} -\lambda & \frac{-1}{\sqrt{1+r^2}} \\ \frac{-1}{(1+r^2)\sqrt{1+r^2}} & -\lambda \end{pmatrix} = \lambda^2 - (\frac{-1}{\sqrt{1+r^2}})(\frac{-1}{(1+r^2)\sqrt{1+r^2}}) = \lambda^2 - \frac{1}{(r^2+1)^2}$$

$$\Rightarrow \lambda = \pm\frac{1}{r^2+1}.$$

Finally, perform the inverse change of variables back to rectangular coordinates and confirm that the result recapitulates the given expression for the principal curvatures,

$$\text{Principal curvatures: } \lambda = \pm\frac{1}{r^2+1} = \pm\frac{1}{x^2+y^2+1}.$$

10. **Consider the parametrized surface**

$$\mathbf{r}(u,v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2\right). \tag{10}$$

Show that

(a) **The coefficients of the first fundamental form are**

$$E = G = (1 + u^2 + v^2)^2, \quad F = 0. \tag{11}$$

(b) **The coefficients of the second fundamental form are**

$$L = 2,\ M = -2,\ N = 0. \tag{12}$$

(c) **The principal curvatures are**

$$\kappa_1 = \frac{2}{(1 + u^2 + v^2)^2}, \quad \kappa_2 = -\frac{2}{(1 + u^2 + v^2)^2}. \tag{13}$$

(a) Compute the first derivatives of $\mathbf{r}$ and their various dot products to subsequently compute the coefficients of the first fundamental form,

$$\begin{aligned}
\mathbf{r}_u &= (1 - u^2 + v^2, 2vu, 2u)\\
\mathbf{r}_v &= (2vu, 1 - v^2 + u^2, -2v)\\
\mathbf{r}_u \cdot \mathbf{r}_u &= (1 - u^2 + v^2)^2 + (2vu)^2 + (2u)^2\\
&= 1 - u^2 + v^2 - u^2 + u^4 - u^2v^2 + v^2 - u^2v^2 + v^4 + 4u^2v^2 + 4u^2\\
&= 1 + 2u^2 + 2v^2 + 2u^2v^2 + u^4 + v^4\\
&= (1 + u^2 + v^2)^2 = E\\
\mathbf{r}_u \cdot \mathbf{r}_v &= (2vu)^2 + (1 - v^2 + u^2)^2 + (-2v)^2\\
&= (1 + u^2 + v^2)^2 = G\\
\mathbf{r}_u \cdot \mathbf{r}_v &= (1 - u^2 + v^2)(2vu) + (1 - v^2 + u^2)(2vu) + (2u)(-2v)\\
&= 2vu(1 - u^2 + v^2 + 1 - v^2 + u^2) - 4uv\\
&= 4vu - 4uv = 0 = F.
\end{aligned}$$

(b) Compute the cross product $\mathbf{r}_u \times \mathbf{r}_v$,

$$\begin{aligned}
\mathbf{r}_u \times \mathbf{r}_v = &\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 - u^2 + v^2 & 2vu & 2u\\ 2vu & 1 - v^2 + u^2 & -2v \end{vmatrix}\\
=&\mathbf{i}[(2vu)(-2v) - (2u)(1 - v^2 + u^2)]\\
&- \mathbf{j}[(1 - u^2 + v^2)(-2v) - (2u)(2vu)]\\
&+ \mathbf{k}[(1 - u^2 + v^2)(1 - v^2 + u^2) - (2vu)(2vu)]\\
=&\mathbf{i}[-4v^2u - 2u + 2uv^2 - 2u^3]\\
&- \mathbf{j}[-2v(1 - u^2 + v^2) - 4u^2v]\\
&+ \mathbf{k}[1 - (v^2 - u^2)^2 - 4u^2v^2]\\
=&\mathbf{i}[-2u(1 + u^2 + v^2)]\\
&+ \mathbf{j}[2v(1 + u^2 + v^2)]\\
&+ \mathbf{k}[1 - u^4 - v^4 - 2u^2v^2].
\end{aligned}$$

Let $w = u^2 + v^2 + 1$, thus

$$\begin{aligned}
&\Rightarrow \mathbf{r}_u \times \mathbf{r}_v = (-2uw, 2vw, 1 - (u^2 + v^2)^2)\\
&\Rightarrow |\mathbf{r}_u \times \mathbf{r}_v|^2 = (-2uw)^2 + (2vw)^2 + (1 - (u^2 + v^2))^2 = w^4\\
&\Rightarrow |\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{(-2uw)^2 + (2vw)^2 + (1 - (u^2 + v^2))^2} = w^2.
\end{aligned}$$

Thus, the normal vector is

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} = \frac{(-2uw, 2vw, 1-(u^2+v^2)^2)}{w^2} = (\frac{-2u}{w}, \frac{2v}{w}, \frac{2}{w} - 1).$$

Next, compute the second derivatives of $\mathbf{r}$,

$$\begin{aligned} \mathbf{r}_{uu} &= (-2u, 2v, 2) \\ \mathbf{r}_{vv} &= (2u, -2v, -2) \\ \mathbf{r}_{uv} &= (2v, 2u, 0). \end{aligned}$$

Next, compute the coefficients of the second fundamental form $L = \mathbf{r}_{uu} \cdot \mathbf{n}$, $M = \mathbf{r}_{uv} \cdot \mathbf{n}$, $N = \mathbf{r}_{vv} \cdot \mathbf{n}$,

$$\begin{aligned} L : (2u, 2v, 2) \cdot \mathbf{n} &= (-2u)(\frac{-2u}{w}) + (2v)(\frac{2v}{w}) + 2(\frac{2}{w} - 1) \\ &= \frac{4u^2}{w} + \frac{4v^2}{w} + \frac{4}{w} - 2 = \frac{4w}{w} - 2 = 2 \\ M : (2v, 2u, 0) \cdot \mathbf{n} &= (2v)(\frac{-2u}{w}) + (2u)(\frac{2v}{w}) + 0(\frac{2}{w} - 1) \\ &= \frac{-4uv + 4uv}{w} = 0 \\ N : (2u, -2v, -2) \cdot \mathbf{n} &= (2u)(\frac{-2u}{w}) + (-2v)(\frac{2v}{w}) + (-2)(\frac{2}{w} - 1) \\ &= \frac{-4u^2}{w} - \frac{4v^2}{w} + 2(1 - \frac{2}{w}) = \frac{-4(w-1) - 4}{w} + 2 = \frac{-4w + 4 - 4}{w} + 2 = -4 + 2 = -2. \end{aligned}$$

(c) Using the above results, compute the principal curvatures as the eigenvalues of the shape operator given by the Weingarten matrix $W = I^{-1}II$,

$$\begin{aligned} I &= \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} (1+u^2+v^2)^2 & 0 \\ 0 & (1+u^2+v^2)^2 \end{bmatrix}; \; I^{-1} = \begin{bmatrix} \frac{1}{(1+u^2+v^2)^2} & 0 \\ 0 & \frac{1}{(1+u^2+v^2)^2} \end{bmatrix} \\ II &= \begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \\ \Rightarrow W = I^{-1}II &= \begin{bmatrix} \frac{1}{(1+u^2+v^2)^2} & 0 \\ 0 & \frac{1}{(1+u^2+v^2)^2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} \frac{2}{(1+u^2+v^2)^2} & 0 \\ 0 & \frac{-2}{(1+u^2+v^2)^2} \end{bmatrix} \end{aligned}$$

Finally, compute the eigenvalues i.e., the principal curvatures, via the determinant of $W - \lambda I$,

$$\begin{aligned} \det(W - \lambda I) &= \det \begin{pmatrix} \frac{2}{(1+u^2+v^2)^2} - \lambda & 0 \\ 0 & \frac{-2}{(1+u^2+v^2)^2} - \lambda \end{pmatrix} \\ &\Rightarrow \lambda = \pm \frac{2}{(1+u^2+v^2)^2} \\ &\Rightarrow \kappa_1 = \frac{2}{(1+u^2+v^2)^2}, \quad \kappa_2 = -\frac{2}{(1+u^2+v^2)^2}. \end{aligned}$$