On the Transformation of Homogeneous Differential Expressions of the Second Degree

Author: E. B. Christoffel (Original work "Ueber die Transformation der homogenen Differentialausdrücke zweiten Grades" published in *Journal für die reine und angewandte Mathematik*, Volume 70, 1869, pp. 46–70)

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Date of Translation: August 2, 2025

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Translator's Note: This document presents a complete English translation of E. B. Christoffel's seminal 1869 paper, *Ueber die Transformation der homogenen Differentialausdrücke zweiten Grades*, originally published in German. I simply copied and pasted the text from a pdf of the original German into Grok. The math had to be manually transcribed into LaTeXwhich is arguably the main value I added through this translation.

I don't speak German. Nevertheless, I am reasonably confident in this translation for a few reasons. First, the precise technical language and direct style of Christoffel's paper heavily employs unambiguous mathematical terms that end up being near-cognates in German and English. Second, I used Google Translate to spot-check the text on a word-by-word and sentence-by-sentence level in places that the first automatic translation pass seemed less accurate. Third, the automatically translated text provides a context that seems to agree with the mathematical expressions which are independent of English and German. As such, I have made minimal stylistic edits to the text.

I strove to exactly reproduce the mathematical expressions of Christoffel's paper to preserve the mathematical and historical context of the original work. This paper introduced what are now known as Christoffel symbols and the covariant derivative, which are fundamental to differential geometry and general relativity. However, as an English language reader can hopefully now more easily appreciate through this translation, the notation Christoffel used to introduce these tools is not the same as what we use today. My emphasis has thus been to accurately render the mathematical notation and equations as they are in the primary source document, ensuring fidelity to Christoffel's original usage and the notation conventions he adhered to when he published his work.

This translation is intended for academic and research purposes, and I kindly request that any use or citation of this work includes proper attribution to the translator.

For inquiries, permissions, or to report an error/translation suggestion please contact Anthony at anthonymcknight000@gmail.com.

Suggested Citation: Mcknight, RA. (2025). Translation of "On the Transformation of Homogeneous Differential Expressions of the Second Degree" by E. B. Christoffel (1869). Retrieved from https://www.anthonymakesvideos.com/files/christoffel.pdf.

On the Transformation of Homogeneous Differential Expressions of the Second Degree

E. B. Christoffel in Zürich

1.

If one introduces into the differential expression

$$F = \sum_{i,k=1}^{n} \omega_{ik} \partial x_i \partial x_k; \quad i, k = 1, 2, \ldots, n,$$

in which the coefficients ω are arbitrary functions of the mutually independent variables $x_1, x_2, \ldots, x_n$, a system of mutually independent functions of the new variables $x'_1, x'_2, \ldots, x'_n$, then F transforms into a new differential expression

$$F' = \sum_{i,k=1}^{n} \omega'_{ik} \partial x'_i \partial x'_k,$$

which is equivalent to the original due to the performed substitution.

Conversely, if the two differential expressions F and F' are given, one can pose the question of which conditions are necessary for one to be transformed into the other, and, if this is possible, which substitutions accomplish the required transformation.

For this investigation, we denote the substitution determinant and the determinants of F and F' as follows:

$$\begin{aligned} \sum \pm \; \frac{\partial x_1}{\partial x'_1} \frac{\partial x_2}{\partial x'_2} \cdots \frac{\partial x_n}{\partial x'_n} &= r, \\ \sum \pm \; \omega_{11} \, \omega_{22} \cdots \omega_{nn} &= E, \\ \sum \pm \; \omega'_{11} \, \omega'_{22} \cdots \omega'_{nn} &= E'. \end{aligned}$$

Then, as is known from algebraic invariant theory, one and only one transformation condition holds:

$$E' = r^2 E,$$

and this would suffice for the present case if the coefficients of F and the elements of r were constant. However, in the present case, additional conditions must be satisfied, since not every system of linear and homogeneous functions of $\partial x'_1, \partial x'_2, \ldots, \partial x'_n$, which, when substituted for $\partial x_1, \partial x_2, \ldots, \partial x_n$, transforms F into F', solves the posed problem, but only those that satisfy the integrability conditions, so that the expressions for $\partial x_1, \partial x_2, \ldots, \partial x_n$ become complete differentials.

If, in general, F is a homogeneous differential expression of arbitrary order, and F' is of the same order, then for the transformation of F into F', since linear substitutions apply to the differentials, the condition equations arising from algebraic invariant theory hold, and the associated forms provide, when the order of F is greater than 2, generally completely determined values for the coefficients in the expressions of the original differentials in terms of the new ones to be introduced. However, these coefficients must still satisfy the associated integrability conditions, where the property of the associated forms, which directly provide not the direct but the transposed substitution, is essentially taken into account (cf. § 10).

Such simplifications do not occur for homogeneous differential expressions of the second degree, since for these, only one invariant and one associated form arise from the algebraic conditions. Therefore, in the following, I will set out the tools by which this exceptional case, so important in various fields, can be resolved, and I note only that for the case where $F = \partial x_1^2 + \partial x_2^2 + \partial x_3^2$, an extensive work by Mr. Lamé exists (*Theory of Curvilinear Coordinates*)[1], while the present investigations were originally motivated by the extension of the problem of developable surfaces to spaces of n dimensions.

2.

In investigating the conditions necessary and sufficient for the possibility of the equation

$$\sum \omega_{ik} \partial x_i \partial x_k = \sum \omega'_{ik} \partial x'_i \partial x'_k, \tag{1}$$

I will restrict myself to the case where the determinants E and E' of these differential expressions are not identically zero. The new variables x' are considered independent, and their differentials are assumed constant.

If one replaces each ∂x_i by $\partial x'_i + \delta x'_i$ and denotes the increments of the original variables corresponding to the differentials $\delta x'_i$ and δx_i by the variable x, then in (1), generally, ∂x must be replaced by $\partial x + \delta x$.

If one then expands the squares and products on both sides, it follows after eliminating identical terms:

$$\sum \omega_{ik} \partial x_i \delta x_k = \sum \omega'_{ik} \partial x'_i \delta x'_k. \tag{2}$$

Comparing the coefficients of $\partial x'_g$, it follows

$$\sum_{ik} \omega_{ik} \frac{\partial x_i}{\partial x'_g} \delta x_k = \sum_k \omega'_{gk} \delta x'_k,$$

and from this

$$\delta x'_h = \sum_{gik} \omega_{ik} \frac{E'_{gh}}{E'} \frac{\partial x_i}{\partial x'_g} \delta x_k, \tag{3}$$

where E'_{gh} denotes, in the known manner, the subdeterminant of E' arising from a change in ω'_{gh} (not ω'_{hg}).

In equation (2), let each x' increase by its differential dx', and in (1), by $\delta x'$, whereby the differentials on the right side remain unchanged; then it follows:

$$\sum \omega_{ik} \partial^2 x_i \delta x_k + \sum \partial \omega_{ik} \partial x_i \delta x_k + \sum \omega_{ik} \partial x_i \partial \delta x_k = \sum \partial \omega'_{ik} \partial x'_i \delta x'_k,$$
$$\sum \delta \omega_{ik} \partial x_i \partial x_k + 2 \sum \omega_{ik} \partial x_i \delta \partial x_k = \sum \delta \omega'_{ik} \partial x'_i \partial x'_k.$$

Dividing the second equation by 2 and subtracting it from the previous one, it follows

$$\sum \omega_{ik} \partial^2 x_i \delta x_k + \sum \partial \omega_{ik} \partial x_i \delta x_k - \frac{1}{2} \sum \delta \omega_{ik} \partial x_i \partial x_k = \sum \partial \omega'_{ik} \partial x'_i \delta x'_k - \frac{1}{2} \sum \delta \omega'_{ik} \partial x'_i \partial x'_k.$$

For better clarity, let

$$\frac{1}{2} \left[\frac{\partial \omega_{gk}}{\partial x_h} + \frac{\partial \omega_{hk}}{\partial x_g} - \frac{\partial \omega_{gh}}{\partial x_k} \right] = \begin{bmatrix} gh \\ k \end{bmatrix} \tag{4}$$

from which

$$\begin{bmatrix} hg \\ k \end{bmatrix} = \begin{bmatrix} gh \\ k \end{bmatrix} \quad \text{and} \quad \frac{\partial \omega_{hk}}{\partial x_g} = \begin{bmatrix} gh \\ k \end{bmatrix} + \begin{bmatrix} gk \\ h \end{bmatrix} \tag{5}$$

[1]Translator Note: Gabriel Lamé (1795-1875). The work referred to is "Leçons sur les coordonnées curvilignes et leurs diverses applications" (1859).

follows, and using the same notation for the transformed form, it follows

(1) $$\sum_{ik} \omega_{ik} \partial^2 x_i \delta x_k + \sum_{ikl} \begin{bmatrix} il \\ k \end{bmatrix} \partial x_i \partial x_l \delta x_k = \sum_{\alpha\beta h} \begin{bmatrix} \alpha\beta \\ h \end{bmatrix} \partial x'_\alpha \partial x'_\beta \delta x'_h.$$

Here, replace $\delta x'_h$ by its value from (3), whereby the right side transforms into

$$\sum_{\alpha\beta h} \begin{bmatrix} \alpha\beta \\ h \end{bmatrix} \partial x'_\alpha \partial x'_\beta \sum_{gik} \omega_{ik} \frac{E'_{gh}}{E'} \frac{\partial x_i}{\partial x'_g} \delta x_k$$

By comparing the coefficients of δx_k, it also follows

$$\sum_i \omega_{ik} \partial^2 x_i + \sum_{il} \begin{bmatrix} il \\ k \end{bmatrix} \partial x_i \partial x_l = \sum_{gi\alpha\beta} \omega_{ik} \frac{\partial x_i}{\partial x'_g} \partial x'_\alpha \partial x'_\beta \sum_h \begin{bmatrix} \alpha\beta \\ h \end{bmatrix}' \frac{E'_{gh}}{E'}.$$

I multiply this equation by $\frac{E_{rk}}{E}$ and sum over the values of k. If one then sets

(7) $$\sum_k = \begin{bmatrix} il \\ k \end{bmatrix} \frac{E_{rk}}{E} = \begin{Bmatrix} il \\ r \end{Bmatrix}$$

from which

(8) $$\begin{Bmatrix} li \\ r \end{Bmatrix} = \begin{Bmatrix} il \\ r \end{Bmatrix}$$

follows, and applies the same notation to the transformed form, it follows

$$\partial^2 x_r + \sum_{il} \begin{Bmatrix} il \\ r \end{Bmatrix} \partial x_i \partial x_l = \sum_{\alpha\beta g} \begin{Bmatrix} \alpha\beta \\ g \end{Bmatrix}' \frac{\partial x_r}{\partial x'_g} \partial x'_\alpha \partial x'_\beta$$

from which it is evident that equation (1) follows backward, and thus for all values of α, β, and r:

(9) $$\frac{\partial^2 x_r}{\partial x'_\alpha \partial x'_\beta} + \sum \begin{Bmatrix} ik \\ r \end{Bmatrix} \frac{\partial x_i}{\partial x'_\alpha} \frac{\partial x_k}{\partial x'_\beta} = \sum_\lambda \begin{Bmatrix} \alpha\beta \\ \lambda \end{Bmatrix}' \frac{\partial x_r}{\partial x'_\lambda}.$$

This equation provides $\frac{n(n+1)}{2}$ equations for each r, thus a total system of $n \cdot \frac{n(n+1)}{2}$ simultaneous partial differential equations to determine the sought substitution. If these are satisfied, the integrability conditions for the linear expressions of the original differentials in terms of the new ones are obviously fulfilled.

These equations simplify considerably if all coefficients of the original form are constant, since in this case, but also only under this condition, all expressions $\begin{Bmatrix} ik \\ r \end{Bmatrix}$ vanish due to (7) and (5). One then obtains for each original variable a system of linear partial differential equations, and for all variables the same system. The same result was found by Mr. Lamé for the case he treated, as mentioned above.

3.

If it is not possible to determine the original variables as functions of the new x' such that equations (9) are satisfied, then the transformation of F into F' is impossible, since these equations necessarily follow if the possibility of this transformation is assumed.

However, if equations (9) are satisfied in some way, the question arises to what extent they justify the conclusion to equation (1), i.e., to the formula $F = F'$. To investigate this, introduce the assumed solution of equations (9) into F, from which

$$F = \sum \omega''_{ik} \partial x'_i \partial x'_k = F'',$$

may follow. Then it is a matter of what relationships exist between the corresponding coefficients of F'' and F'.

Now, equations (9) are nothing but an identical reformulation of equation (1), from which they follow and which they also imply backward. However, if one starts from $F = F''$ instead of $F = F'$, the calculation in § 2 yields, instead of the expression on the right side of (1), the new expression

$$\sum_{\alpha\beta h} \begin{bmatrix} \alpha\beta \\ h \end{bmatrix}'' \partial x'_\alpha \partial x'_\beta \delta x'_h,$$

and this must equal the original. Thus, it generally follows

$$\begin{bmatrix} \alpha\beta \\ h \end{bmatrix}'' = \begin{bmatrix} \alpha\beta \\ h \end{bmatrix}'$$

from which, due to (5),

$$\frac{\partial \omega''_{hk}}{\partial x'_g} = \frac{\partial \omega'_{hk}}{\partial x'_g}$$

follows for all values of g, h, and k. The corresponding coefficients of F'' and F' thus differ only by additive constants. For these constants to vanish, it suffices that $F'' = F'$ holds at least once, in other words, that the solution of equations (9) satisfies the transformation relations arising from equation (1)

$$\sum_{ik} \omega_{ik} \frac{\partial x_i}{\partial x'_\alpha} \frac{\partial x_k}{\partial x'_\beta} = \omega'_{\alpha\beta} \tag{10}$$

for some specific set of values of the new variables.

We thus have the theorem:

If it is possible to determine the original variables $x_1, x_2, \ldots, x_n$ as functions of the new variables $x'_1, x'_2, \ldots, x'_n$ such that the system of equations (9) is satisfied, and these functions are further subjected to the initial condition that they, along with their first derivatives, satisfy the transformation relations contained in the equation $F = F'$ for some specific set of values of the new variables, then these transformation relations, and with them the equation $F = F'$, hold for all values of the new variables.

It follows that the transformation relations contained in (1) are rendered superfluous by equations (9), up to the initial condition derived from them, as soon as the latter are compatible with each other.

The system of equations (9) thus contains, assuming its possibility, all the necessary and sufficient conditions for the transformation of F into F', up to the mentioned initial condition, and thus simultaneously replaces the algebraic transformation relations arising from (1) and the associated integrability conditions.

4.

For the possibility of equations (9), new integrability conditions are required, to whose derivation we now proceed. To simultaneously demonstrate that these do not imply equations (9) backward, we consider instead of the latter the system contained in one of the following equations:

$$\left\{ \begin{aligned} \frac{\partial^2 x_r}{\partial x'_\alpha \partial x'_\beta} + \sum_{gh} \begin{Bmatrix} gh \\ r \end{Bmatrix} u^g_\alpha u^h_\beta &= \sum_\lambda \begin{Bmatrix} \alpha\beta \\ \lambda \end{Bmatrix}' u^r_\lambda + \begin{pmatrix} r \\ \alpha\beta \end{pmatrix}, \\ \frac{\partial^2 x_r}{\partial x'_\alpha \partial x'_\gamma} + \sum_{gi} \begin{Bmatrix} gi \\ r \end{Bmatrix} u^g_\alpha u^i_\gamma &= \sum_\lambda \begin{Bmatrix} \alpha\gamma \\ \lambda \end{Bmatrix}' u^r_\lambda + \begin{pmatrix} r \\ \alpha\gamma \end{pmatrix}, \end{aligned} \right. \tag{9'.}$$

where, as generally in the following, the first derivatives

$$\frac{\partial x_i}{\partial x_\alpha} = u^i_\alpha$$

are set, while we retain the usual notation for higher derivatives. From the above, the system (9) is obtained by setting all quantities $\begin{pmatrix} r \\ \alpha\beta \end{pmatrix}$ equal to 0.

The integrability conditions in question are obtained by differentiating the first equation (9′.) with respect to x'_γ, the second with respect to x'_β, and taking the difference, whereby the third derivative of x_r cancels out. Noting that the terms containing a second derivative with respect to x'_β and x'_γ also cancel, it follows:

$$\begin{aligned}
&\sum_{ghi}\left[\frac{\partial \begin{Bmatrix} gh \\ r \end{Bmatrix}}{\partial x_i} - \frac{\partial \begin{Bmatrix} gi \\ r \end{Bmatrix}}{\partial x_h}\right] u^g_\alpha u^h_\beta u^i_\gamma + \sum_{ph}\begin{Bmatrix} ph \\ r \end{Bmatrix}\frac{\partial^2 x_p}{\partial x'_\alpha \partial x'_\gamma} u^h_\beta - \sum_{pi}\begin{Bmatrix} pi \\ r \end{Bmatrix}\frac{\partial^2 x_p}{\partial x'_\alpha \partial x'_\beta} u^i_\gamma \\
&= \sum_\lambda \left[\frac{\partial \begin{Bmatrix} \alpha\beta \\ \lambda \end{Bmatrix}'}{\partial x_\gamma} - \frac{\partial \begin{Bmatrix} \alpha\gamma \\ \lambda \end{Bmatrix}'}{\partial x_\beta}\right] u^r_\lambda + \sum_\lambda \begin{Bmatrix} \alpha\beta \\ \lambda \end{Bmatrix}' \frac{\partial^2 x_r}{\partial x'_\lambda \partial x'_\gamma} - \sum_\lambda \begin{Bmatrix} \alpha\gamma \\ \lambda \end{Bmatrix}' \frac{\partial^2 x_r}{\partial x'_\lambda \partial x'_\beta} \\
&\qquad + \frac{\partial \begin{pmatrix} r \\ \alpha\beta \end{pmatrix}}{\partial x'_\gamma} - \frac{\partial \begin{pmatrix} r \\ \alpha\gamma \end{pmatrix}}{\partial x'_\beta}.
\end{aligned}$$

From this, it follows first that the integrability conditions for the system (9′.) will be exactly the same as for the original equations (9), as soon as the quantities $\begin{pmatrix} r \\ \alpha\beta \end{pmatrix}$ satisfy the following system of equations

$$(11)\qquad \left\{\begin{aligned}
&\sum_{ph}\begin{Bmatrix} ph \\ r \end{Bmatrix}\begin{pmatrix} p \\ \alpha\gamma \end{pmatrix} u^h_\beta - \sum_{pi}\begin{Bmatrix} pi \\ r \end{Bmatrix}\begin{pmatrix} p \\ \alpha\beta \end{pmatrix} u^i_\gamma \\
&= \sum_\lambda \begin{Bmatrix} \alpha\beta \\ \lambda \end{Bmatrix}'\begin{pmatrix} r \\ \lambda\gamma \end{pmatrix} u - \sum_\lambda \begin{Bmatrix} \alpha\gamma \\ \lambda \end{Bmatrix}'\begin{pmatrix} r \\ \lambda\beta \end{pmatrix} + \frac{\partial \begin{pmatrix} r \\ \alpha\beta \end{pmatrix}}{\partial x'_\gamma} - \frac{\partial \begin{pmatrix} r \\ \alpha\gamma \end{pmatrix}}{\partial x'_\beta}
\end{aligned}\right. ,$$

which can easily be put into a symmetric form.

I now assume that this condition is satisfied for all values of r, α, β, γ. If one then substitutes the values of the second derivatives from (9′.) into the previous equation, it follows:

$$\begin{aligned}
&\sum_{ghi}\left[\frac{\partial \begin{Bmatrix} gh \\ r \end{Bmatrix}}{\partial x_i} - \frac{\partial \begin{Bmatrix} gi \\ r \end{Bmatrix}}{\partial x_h}\right] u^g_\alpha u^h_\beta u^i_\gamma \\
&+ \sum_{ph}\begin{Bmatrix} ph \\ r \end{Bmatrix} u^h_\beta \left(\sum_\lambda \begin{Bmatrix} \alpha\gamma \\ \lambda \end{Bmatrix}' u^p_\lambda - \sum_{gi}\begin{Bmatrix} gi \\ p \end{Bmatrix} u^g_\alpha u^i_\gamma\right) - \sum_{pi}\begin{Bmatrix} pi \\ r \end{Bmatrix} u^i_\gamma \left(\sum_\lambda \begin{Bmatrix} \alpha\beta \\ \lambda \end{Bmatrix}' u^p_\lambda - \sum_{gh}\begin{Bmatrix} gh \\ p \end{Bmatrix} u^g_\alpha u^h_\beta\right) \\
&= \sum_\lambda \left[\frac{\partial \begin{Bmatrix} \alpha\beta \\ \lambda \end{Bmatrix}'}{\partial x'_\gamma} - \frac{\partial \begin{Bmatrix} \alpha\gamma \\ \lambda \end{Bmatrix}'}{\partial x'_\beta}\right] u^r_\lambda \\
&+ \sum_\lambda \begin{Bmatrix} \alpha\beta \\ \lambda \end{Bmatrix}' \left(\sum_\mu \begin{Bmatrix} \gamma\lambda \\ \mu \end{Bmatrix}' u^r_\mu - \sum_{pi}\begin{Bmatrix} pi \\ r \end{Bmatrix} u^p_\lambda u^i_\lambda\right) - \sum_\lambda \begin{Bmatrix} \alpha\gamma \\ \lambda \end{Bmatrix}' \left(\sum_\mu \begin{Bmatrix} \beta\lambda \\ \mu \end{Bmatrix}' u^r_\mu - \sum_{ph}\begin{Bmatrix} ph \\ r \end{Bmatrix} u^p_\lambda u^h_\beta\right).
\end{aligned}$$

Here, all terms containing the product of two derivatives u cancel out; combining the remaining terms by swapping β with γ on the right, it follows:

$$(12)\qquad \left\{\begin{aligned}
&\sum_{ghi}\left(\frac{\partial \begin{Bmatrix} gh \\ r \end{Bmatrix}}{\partial x_i} - \frac{\partial \begin{Bmatrix} gi \\ r \end{Bmatrix}}{\partial x_h} + \sum_p \left[\begin{Bmatrix} gh \\ p \end{Bmatrix}\begin{Bmatrix} pi \\ r \end{Bmatrix} - \begin{Bmatrix} gi \\ p \end{Bmatrix}\begin{Bmatrix} ph \\ r \end{Bmatrix}\right]\right) u^g_\alpha u^h_\beta u^i_\gamma \\
&= \sum_\lambda \left(\frac{\partial \begin{Bmatrix} \alpha\beta \\ \lambda \end{Bmatrix}'}{\partial x_\gamma} - \frac{\partial \begin{Bmatrix} \alpha\gamma \\ \lambda \end{Bmatrix}'}{\partial x_\beta} + \sum_\mu \left[\begin{Bmatrix} \alpha\beta \\ \mu \end{Bmatrix}'\begin{Bmatrix} \mu\gamma \\ \lambda \end{Bmatrix}' - \begin{Bmatrix} \alpha\gamma \\ \mu \end{Bmatrix}'\begin{Bmatrix} \mu\beta \\ \lambda \end{Bmatrix}'\right]\right) u^r_\lambda
\end{aligned}\right.$$

We thus have the following result:

The system of equations resulting from (12), when all values of r, β, γ from 1 to n are substituted, contains all the integrability conditions necessary for the possibility of the system (9). However, it cannot replace equations (9), as it does not imply them backward but rather the more general equations (9′.), in which the independent terms f^i_{gr} satisfy the system of equations (11).

5.

The equations (12) must be replaced for further investigation by another equivalent system. For this purpose, I multiply by $\omega_{rk} u^k_\delta$ and sum over all values of k and r. It is clear that the resulting equations lead back to the system (12) if, after the designated summation is performed, one multiplies by $\frac{\partial x'_\delta}{\partial x_l} \frac{E_{\varrho l}}{E}$, then sums first over δ, then over α, and finally sets $\varrho = r$.

In the specified operation, equation (12), since due to (10)

$$\sum_{rk} \omega_{rk} u^r_\lambda u^k_\delta = \omega'_{\lambda\delta}$$

first transforms into

$$\sum_{ghik} u^g_\alpha u^h_\beta u^i_\gamma u^k_\delta \sum_r \omega_{rk} \left(\frac{\partial \begin{Bmatrix} gh \\ r \end{Bmatrix}}{\partial x_i} - \frac{\partial \begin{Bmatrix} gi \\ r \end{Bmatrix}}{\partial x_h} + \sum_p \left[\begin{Bmatrix} gh \\ p \end{Bmatrix} \begin{Bmatrix} pi \\ r \end{Bmatrix} - \begin{Bmatrix} gi \\ p \end{Bmatrix} \begin{Bmatrix} ph \\ r \end{Bmatrix} \right] \right)$$
$$= \sum_\lambda \omega'_{\lambda\delta} \left(\frac{\partial \begin{Bmatrix} \alpha\beta \\ \lambda \end{Bmatrix}'}{\partial x'_\gamma} - \frac{\partial \begin{Bmatrix} \alpha\gamma \\ \lambda \end{Bmatrix}}{\partial x'_\beta} + \sum_\mu \left[\begin{Bmatrix} \alpha\beta \\ \mu \end{Bmatrix} \begin{Bmatrix} \mu\gamma \\ \lambda \end{Bmatrix}' - \begin{Bmatrix} \alpha\gamma \\ \mu \end{Bmatrix}' \begin{Bmatrix} \mu\beta \\ \lambda \end{Bmatrix}' \right] \right)$$

so that the remaining reductions are the same for both sides of the equation. Now, according to equation (7),

$$\sum_r \omega_{rk} \begin{Bmatrix} gh \\ r \end{Bmatrix} = \begin{bmatrix} gh \\ k \end{bmatrix},$$

thus the sum taken over r on the left side becomes

$$= \frac{\partial \begin{bmatrix} gh \\ k \end{bmatrix}}{\partial x_i} - \sum_p \begin{Bmatrix} gh \\ p \end{Bmatrix} \frac{\partial \omega_{pk}}{\partial x_i} - \frac{\partial \begin{bmatrix} gi \\ k \end{bmatrix}}{\partial x_h} + \sum_p \begin{Bmatrix} gi \\ p \end{Bmatrix} \frac{\partial \omega_{pk}}{\partial x_h} + \sum_p \left(\begin{Bmatrix} gh \\ p \end{Bmatrix} \begin{bmatrix} pi \\ k \end{bmatrix} - \begin{Bmatrix} gi \\ p \end{Bmatrix} \begin{bmatrix} ph \\ k \end{bmatrix} \right),$$

where in the two sums containing derivatives of ω, p is set in place of r. Further, according to (5),

$$\frac{\partial \omega_{pk}}{\partial x_i} = \begin{bmatrix} ip \\ k \end{bmatrix} + \begin{bmatrix} ik \\ p \end{bmatrix}, \quad \frac{\partial \omega_{pk}}{\partial x_h} = \begin{bmatrix} hp \\ k \end{bmatrix} + \begin{bmatrix} hk \\ p \end{bmatrix}$$

thus

$$\begin{bmatrix} pi \\ k \end{bmatrix} - \frac{\partial \omega_{pk}}{\partial x_i} = - \begin{bmatrix} ik \\ p \end{bmatrix}, \quad \frac{\partial \omega_{pk}}{\partial x_h} - \begin{bmatrix} ph \\ k \end{bmatrix} = \begin{bmatrix} hk \\ p \end{bmatrix},$$

and the sum in question becomes

$$= \frac{\partial \begin{bmatrix} gh \\ k \end{bmatrix}}{\partial x_i} - \frac{\partial \begin{bmatrix} gi \\ k \end{bmatrix}}{x_h} + \sum_p \left(\begin{Bmatrix} gi \\ p \end{Bmatrix} \begin{bmatrix} hk \\ p \end{bmatrix} - \begin{Bmatrix} gh \\ p \end{Bmatrix} \begin{bmatrix} ik \\ p \end{bmatrix} \right).$$

I denote this expression henceforth by $(gkhi)$, so that, when the quantities $\begin{Bmatrix} gi \\ p \end{Bmatrix}, \begin{Bmatrix} gh \\ p \end{Bmatrix}$ are eliminated using equations (7), it becomes

$$(13) \qquad (gkhi) = \frac{\partial \begin{bmatrix} gh \\ k \end{bmatrix}}{\partial x_i} - \frac{\partial \begin{bmatrix} gi \\ k \end{bmatrix}}{x_h} + \sum_{\alpha\beta} \frac{E_{\alpha\beta}}{E} \left(\begin{bmatrix} gi \\ \alpha \end{bmatrix} \begin{bmatrix} hk \\ \beta \end{bmatrix} - \begin{bmatrix} gh \\ \alpha \end{bmatrix} \begin{bmatrix} ik \\ \beta \end{bmatrix} \right),$$

or, when the derivatives are carried out,

$$(14)\quad \begin{cases} (gkhi) = \dfrac{1}{2}\left(\dfrac{\partial^2\omega_{gi}}{\partial x_h \partial x_k} + \dfrac{\partial^2\omega_{hk}}{\partial x_g \partial x_i} - \dfrac{\partial^2\omega_{gh}}{\partial x_i \partial x_k} - \dfrac{\partial^2\omega_{ik}}{\partial x_g \partial x_h}\right) \\ \qquad + \sum\limits_{\alpha\beta} \frac{E_{\alpha\beta}}{E}\left(\begin{Bmatrix} gi \\ \alpha \end{Bmatrix}\begin{bmatrix} hk \\ \beta \end{bmatrix} - \begin{Bmatrix} gh \\ \alpha \end{Bmatrix}\begin{bmatrix} ik \\ \beta \end{bmatrix}\right) \end{cases}$$

Using this notation, which we also apply to the transformed form, the integrability conditions (12) take the following form:

$$(15)\quad (\alpha\delta\beta\gamma)' = \sum_{ghik} (gkhi) u_\alpha^g u_\beta^h u_\gamma^i u_\delta^k.$$

For this system of equations, the same result holds as for equations (12), namely that they form the complete system of all integrability conditions associated with (9), but they do not imply equations (9), but rather the more general system (9′.), assuming the relations (11) hold.

The coefficients in the above system of formulas have the following properties. Swapping i with h in (13), it follows

$$(16a)\quad (gkih) = -(gkhi).$$

Swapping g with k in (14), it follows

$$(16b)\quad (kghi) = -(gkhi).$$

Swapping g with i and simultaneously k with h in (14), since $E_{\beta\alpha} = E_{\alpha\beta}$, nothing changes on the right side, and it follows

$$(ihkg) = (gkhi),$$

or

$$(16c)\quad (higk) = (gkhi).$$

Finally, it is obtained directly

$$(16d)\quad (gkhi) + (ghik) + (gikh) = 0,$$

noting that these four formulas reduce to one of the previous ones as soon as two of the indices g, h, i, k are equal.

Using these four formulas, the number of essentially distinct relations contained in (15) can be easily determined. First, due to (16a) and (16b), all cases where $\alpha = \delta$ or $\beta = \gamma$ must be excluded, as well as those that follow by swapping α with δ or β with γ. Thus, for $\alpha\delta$ as well as for $\beta\gamma$, only combinations without repetition of the numbers $1, 2, \ldots, n$ are to be taken, whose number is $\frac{n(n-1)}{2} = n_2$.

Now, the expressions $(\alpha\delta\beta\gamma)'$ fall into three groups: the first consists of n_2 expressions corresponding to the assumption $\alpha\delta = \beta\gamma$, the second consists of $\frac{n_2(n_2-1)}{2}$ expressions where $\alpha\delta \neq \beta\gamma$, while the third group of equally many expressions, which arise from the second by swapping $\alpha\delta$ with $\beta\gamma$, is excluded due to (16c). Thus, due to (16a), (16b), and (16c), only $\frac{n_2(n_2-1)}{2}$ expressions $(\alpha\delta\beta\gamma)'$ remain.

However, since each group $\alpha\delta\beta\gamma$, in which no number appears more than once, provides one equation (16d), a total of

$$n_4 = \frac{1}{24} n(n-1)(n-2)(n-3)$$

of the aforementioned expressions can be expressed by others, leaving only

$$\frac{n_2(n_2+1)}{2} - n_4 = \frac{n^2(n^2-1)}{12}$$

essentially distinct expressions $(\alpha\delta\beta\gamma)'$. This is thus the number of equations (15), insofar as they do not follow from one another.

The number of equations (10) is $\frac{n(n+1)}{2}$; adding this to the above number, we obtain

$$\frac{n(n+1)}{2}+\frac{n^2(n^2-1)}{12}=n^2+n+\frac{(n+2)(n+1)n(n-3)}{12}$$

and thus, for $n=2$, the number of equations (10) and (15) is less than the number n^2+n of unknowns appearing in them, namely the original variables and their first derivatives; for $n=3$, it is equal to this number, and for $n>3$, it is greater than the latter.

6.

Just as equations (10) serve as transformation relations for equation (1) or (2), so too are equations (15), as their form indicates, the transformation relations for a form of the fourth order. Before we proceed to establish this form, we will demonstrate how, in general, from the transformation relations associated with any form, one can obtain, using equations (9), the transformation relations for a new form whose order is one greater than that of the original form.

Let

$$G_\mu=\sum_{i_1\dots i_\mu}(i_1i_2\dots i_\mu)\underset{1}{\partial}x_{i_1}\underset{2}{\partial}x_{i_2}\dots\underset{\mu}{\partial}x_{i_\mu},$$

be a form derived from the coefficients of F, linear in the differentials $\underset{1}{\partial}x,\underset{2}{\partial}x,\dots,\underset{\mu}{\partial}x$ μ-times, and G'_μ its transformed form, thus generally

$$(\alpha_1\alpha_2\dots\alpha_\mu)'=\sum_{i_1\dots i_\mu}(i_1i_2\dots i_\mu)u^{i_1}_{\alpha_1}u^{i_2}_{\alpha_2}\dots u^{i_\mu}_{\alpha_\mu},$$

where, as before,

$$u^i_\alpha=\frac{\partial x_i}{\partial x'_\alpha}$$

is set. Differentiating with respect to x'_α, it follows

$$\begin{aligned}\frac{\partial(\alpha_1\alpha_2\dots\alpha_p)'}{\partial x'_\alpha}&=\sum_{i\dots 1_\mu}\frac{\partial(i_1i_2\dots i_\mu)}{\partial x_i}u^i_\alpha u^{i_1}_{\alpha_1}\dots u^{i_\mu}_{\alpha_\mu}+P,\\ P&=\sum_{\lambda i_2\dots i_\mu}(\lambda i_2\dots i_\mu)\frac{\partial^2x_\lambda}{\partial x'_\alpha\partial x'_{\alpha_1}}u^{i_2}_{\alpha_2}\dots u^{i_\mu}_{\alpha_\mu}\\ &\quad+\sum_{i_1\lambda\dots i_\mu}(i_1\lambda\dots i_\mu)u^{i_1}_{\alpha_1}\frac{\partial^2x_\lambda}{\partial x'_\alpha\partial x'_{\alpha_2}}\dots u^{i_\mu}_{\alpha_\mu}+\cdots.\end{aligned}$$

Now, substituting the values of the second derivatives

$$\frac{\partial^2x_\lambda}{\partial x'_\alpha\partial x'_{\alpha_s}}=\sum_r\begin{Bmatrix}\alpha\alpha_s\\ r\end{Bmatrix}'u^\lambda_r-\sum_{ii_s}\begin{Bmatrix}ii_s\\ \lambda\end{Bmatrix}u^i_\alpha u^{i_s}_{\alpha_s}$$

into P, one obtains for P a difference $U-V$, where

$$\begin{aligned}U&=\sum_r\begin{Bmatrix}\alpha\alpha_1\\ r\end{Bmatrix}'\sum_{\lambda i_2\dots i_\mu}(\lambda i_2\dots i_\mu)u^\lambda_r u^{i_2}_{\alpha_2}\dots u^{i_\mu}_{\alpha_\mu}\\ &\quad+\sum_r\begin{Bmatrix}\alpha\alpha_2\\ r\end{Bmatrix}'\sum_{i_1\lambda\dots i_\mu}(i_1\lambda\dots i_\mu)u^{i_1}_{\alpha_1}u^\lambda_r\dots u^{i_\mu}_{\alpha_\mu}+\cdots\\ &=\sum_r\left[\begin{Bmatrix}\alpha\alpha_1\\ r\end{Bmatrix}'(r\alpha_2\dots\alpha_\mu)'+\begin{Bmatrix}\alpha\alpha_2\\ r\end{Bmatrix}'(\alpha_1r\dots\alpha_\mu)'+\cdots\right]\end{aligned}$$

and

$$V=\sum_{i\ldots i_\mu} u^i_\alpha u^{i_1}_{\alpha_1}\ldots u^{i_\mu}_{\alpha_\mu}\sum_\lambda\left[\begin{Bmatrix} ii_1\\ \lambda\end{Bmatrix}(\lambda i_2\ldots i_\mu)+\begin{Bmatrix} ii_2\\ \lambda\end{Bmatrix}(i_1\lambda\ldots i_\mu)+\cdots\right]$$

Substituting the value of P thus determined and bringing U to the left side of the equation, while placing everything on the right under a summation sign, the following result is obtained: *Assuming all integrability conditions are satisfied, from any complete system of transformation relations of order* p

(17a) $$(\alpha_1\alpha_2\ldots\alpha_\mu)'=\sum_{i_1\ldots i_\mu}(i_1i_2\ldots i_\mu)u^{i_1}_{\alpha_1}u^{i_2}_{\alpha_2}\ldots u^{i_\mu}_{\alpha_\mu}$$

one obtains a new complete system of transformation relations of order $(\mu+1)$

(17b) $$(\alpha_1\alpha_2\ldots\alpha_\mu)'=\sum_{i_1\ldots i_\mu}(ii_1\ldots i_\mu)u^i_\alpha u^{i_1}_{\alpha_1}\ldots u^{i_\mu}_{\alpha_\mu}$$

when

(17c) $$(ii_1\ldots i_\mu)=\frac{\partial(i_1i_2\ldots i_\mu)}{\partial x_i}-\sum_\lambda\left[\begin{Bmatrix} ii_1\\ \lambda\end{Bmatrix}(\lambda i_2\ldots i_\mu)+\begin{Bmatrix} ii_2\\ \lambda\end{Bmatrix}(i_1\lambda\ldots i_\mu)+\cdots\right]$$

using this notation also for the transformed form.

Using this theorem, from an equation $G_\mu=G'_\mu$, one can derive a series of similar equations $G_{\mu+1}=G'_{\mu+1}, G_{\mu+2}=G'_{\mu+2},\ldots$ etc. until one arrives at identical relations or those composed of earlier ones.

The first case occurs with the function F itself. Namely, taking $(i_1i_2)=\omega_{i_1i_2}$, it follows

$$\begin{aligned}(ii_1i_2)&=\frac{\partial\omega_{i_1i_2}}{\partial x_i}-\sum_\lambda\left[\begin{Bmatrix} ii_1\\ \lambda\end{Bmatrix}\omega_{\lambda i_2}+\begin{Bmatrix} ii_2\\ \lambda\end{Bmatrix}\omega_{i_1\lambda}\right]\\ &=\frac{\partial\omega_{i_1i_2}}{\partial x_i}-\begin{bmatrix} ii_1\\ i_2\end{bmatrix}-\begin{bmatrix} ii_2\\ i_1\end{bmatrix}\end{aligned}$$

according to a formula used in § 5, and this is always equal to zero due to equation (5).

7.

We now proceed to the representation of the homogeneous form G_4, to which equations (15) belong as transformation relations. These are

(18) $$(\alpha\alpha_1\alpha_2\alpha_3)'=\sum_{i\ldots i_3}(ii_1i_2i_3)\frac{\partial x_i}{\partial x'_\alpha}\frac{\partial x_{i_1}}{\partial x'_{\alpha_1}}\frac{\partial x_{i_2}}{\partial x'_{\alpha_2}}\frac{\partial x_{i_3}}{\partial x'_{\alpha_3}},$$

and due to (16),

(19) $$\begin{cases}(ii_1i_3i_2)=-(ii_1i_2i_3),\quad (i_1ii_2i_3)=-(ii_1i_2i_3),\quad (i_2i_3ii_1)=(ii_1i_2i_3),\\ \qquad (ii_1i_2i_3)+(ii_2i_3i_1)+(ii_3i_1i_2)=0.\end{cases}$$

We multiply equation (18) by $\partial x'_\alpha\delta x'_{\alpha_1}Dx'_{\alpha_2}\Delta x'_{\alpha_3}$, where ∂,δ,D,Δ denote different differentiation symbols, and set

$$\sum_{i\ldots i_3}(ii_1i_2i_3)\partial x_i\delta x_{i_1}Dx_{i_2}\Delta x_{i_3}=G_4;$$

then, from (18) by summation over all values of α, it follows

$$G'_4=G_4.$$

Thus, G_4 is a quadrilinear form of the variable systems $\partial x,\delta x,Dx,\Delta x$, all subject to the same linear substitution. However, due to the properties of its coefficients, this form is highly specific.

Swapping i with i_1, which changes nothing since both run through the same values, and then replacing $(i_1 i i_2 i_3)$ by $-(i i_1 i_2 i_3)$, it follows that G_4 only changes its sign when ∂ is swapped with δ. Taking for $i i_1$ only combinations without repetition, it follows

$$G_4 = \sum (i i_1 i_3 i_4)(\partial x_i \delta x_{i_1} - \partial x_{i_1} \delta x_{i_1}) D x_{i_3} \Delta x_{i_4}.$$

The same applies to D and δ if i_3 and i_4 are swapped, and it follows

$$G_4 = \sum (i i_1 i_2 i_3)(\partial x_i \delta x_{i_1} - \partial x_{i_1} \delta x_{i_1})(D x_{i_2} \Delta x_{i_3} - D x_{i_3} \Delta x_{i_2}) \tag{20}$$

where the prime on the summation sign indicates that only the $\frac{n(n-1)}{2}$ combinations without repetition of the numbers $1, 2, \ldots, n$ are to be taken for both $i i_1$ and $i_2 i_3$.

Swapping i with i_2, i_1 with i_3, and applying the equation $(i_2 i_3 i i_1) = (i i_1 i_2 i_3$, it follows that G_4 remains unchanged when $\partial x_i \delta x_k - \partial x_k \delta x_i$ is swapped with $D x_i \Delta x_k - D x_k \Delta x_i$. Nevertheless, one may not set $D = \partial$ and $\Delta = \delta$, or either of them, because then the coefficients $(i i_1 i_2 i_3)$ and $(i_2 i_1 i i_3)$, or equivalently $(i i_1 i_2 i_3)$ and $(i i_3 i_2 i_1)$, would no longer remain separate but would be combined into their sum, so that not all transformation relations (18) could be derived from $G_4 = G_4'$.

The latter also follow from the forms

$$H_4 = \sum (i i_2 i_3 i_1) \partial x_i \delta x_{i_1} D x_{i_2} \Delta x_{i_3} = \sum (i i_1 i_2 i_3) \partial x_i D x_{i_1} \Delta x_{i_2} \delta x_{i_3},$$
$$J_4 = \sum (i i_3 i_1 i_2) \partial x_i \delta x_{i_1} D x_{i_2} \Delta x_{i_3} = \sum (i i_1 i_2 i_3) \partial x_i \Delta x_{i_1} \delta x_{i_2} D x_{i_3},$$

which arise from G_4 by cyclic permutation of δ, D, Δ, and due to (19), it is identically

$$G_4 + H_4 + J_4 = 0.$$

8.

I now assume that, following § 6, new expressions $(i_1 i_2 i_3 i_4)$ have been derived from the coefficients $(i i_1 i_2 i_3 i_4)$ of G_4, and using these, a fivefold linear form G_5 in the differentials $\partial x, \underset{1}{\partial x}, \underset{2}{\partial x}, \underset{3}{\partial x}, \underset{4}{\partial x}$ has been constructed. Similarly, from G_5, a sixfold linear form G_6, and from this a sevenfold linear form G_7, and so on, until forms are reached whose coefficients either vanish or reduce to earlier forms. Then, the system of equations

$$F = F', \quad G_4 = G_4', \quad G_5 = G_5', \ldots, \tag{A}$$

is obtained, and it follows from the above that for the possibility of the first of these equations, the possibility of all the others is necessary.

For the equation $F = F'$ to be possible, it is necessary and sufficient that the corresponding transformation relations

$$(\alpha_1 \alpha_2)' = \sum_{i_1, i_2} (i_1 i_2) u_{\alpha_1}^{i_1} u_{\alpha_2}^{i_2} \tag{F}$$

for $(ik) = \omega_{ik}$, and simultaneously the integrability conditions

$$\frac{\partial u_\alpha^i}{\partial x_\beta'} = \frac{\partial u_\beta^i}{\partial x_\alpha'} \tag{B}$$

are satisfied. For if both systems of equations are satisfied, there exist n functions $v_1, v_2, \ldots, v_n$ such that equations (F) are satisfied by the general assumption

$$u_\alpha^i = \frac{\partial v_i}{\partial x_\alpha'}$$

and the substitution $x_1 = v_1, x_2 = v_2, \ldots, x_n = v_n$ then accomplishes the required transformation of F into F'.

For the equation $G_4 = G'_4$ to be possible, the transformation relations

$$(G_4) \qquad (\alpha_1\alpha_2\alpha_3\alpha_4)' = \sum_{i_1\ldots i_4} (i_1 i_2 i_3 i_4) u^{i_1}_{\alpha_1} u^{i_2}_{\alpha_2} u^{i_3}_{\alpha_3} u^{i_4}_{\alpha_4}$$

and the integrability conditions (B) must be satisfied.

Similarly, these latter, along with the transformation relations

$$(G_5) \qquad (\alpha\alpha_1 \ldots \alpha_4)' = \sum_{i_1\ldots i_4} (i i_1 \ldots i_4) u^{i}_{\alpha} u^{i_1}_{\alpha_1} \ldots u^{i_4}_{\alpha_4}$$

must be satisfied for the equation $G_5 = G'_5$ to be possible, and so forth.

This established, the transformation of F into F' would already be impossible if the various systems of transformation relations (F), (G_4), (G_5), (G_6), etc., were algebraically incompatible, regardless of the necessarily added integrability conditions (B).

I therefore assume, disregarding equations (B) entirely, that the various systems of transformation relations, whose derivation and sequence have been established above, are algebraically satisfied; then the question is whether the integrability conditions (B) by means of which all other equations (A) follow from $F = F'$ are superfluous or not.

9.

To resolve this question, which is the core of the entire transformation problem, I will, to avoid unnecessary calculations, first present the necessary conclusions under a specific assumption, which is not to be strictly maintained, and then proceed to treat the question in the form in which it must be posed.

I thus assume, 1) that the unknowns x and u are completely and without contradiction determined by the system of equations (G_4), and 2) that these values of the unknowns also satisfy the subsequent system of equations (G_5).

Taking the derivative of equations (G_4) with respect to x'_α, a system of equations follows

$$(G'_4) \qquad \begin{cases} \dfrac{\partial(\alpha_1\alpha_2\alpha_3\alpha_4)'}{\partial x'_\alpha} = \displaystyle\sum_{i_1\ldots i_4} \dfrac{\partial(i_1 i_2 i_3 i_4)}{\partial x_i} \dfrac{\partial x_i}{\partial x'_\alpha} u^{i_1}_{\alpha_1} \ldots u^{i_4}_{\alpha_4} + \Pi, \\ \Pi = \displaystyle\sum_{\lambda i_2 i_3 i_4} (\lambda i_2 i_3 i_4) \dfrac{\partial u^{\lambda}_{\alpha_1}}{\partial x'_\alpha} u^{i_2}_{\alpha_2} u^{i_3}_{\alpha_3} u^{i_4}_{\alpha_4}, \\ \qquad + \displaystyle\sum_{i_1 \lambda i_3 i_4} (i_1 \lambda i_3 i_4) u^{i_1}_{\alpha_1} \dfrac{\partial u^{\lambda}_{\alpha_2}}{\partial x'_\alpha} u^{i_3}_{\alpha_3} u^{i_4}_{\alpha_4} + \cdots . \end{cases}$$

Regarding this system, it must first be established what follows from our first assumption, namely that it completely determines the values of all derivatives

$$\frac{\partial x_i}{\partial x'_\alpha}, \quad \frac{\partial u^{\lambda}_{\alpha_s}}{\partial x'_\alpha}$$

i.e., that the system (G'_4) contains a number of equations equal to the number of these unknowns, which are independent of each other with respect to those, and no equation of this system contradicts others.

Now set

$$\frac{\partial x_i}{\partial x'_\alpha} - u^i_\alpha = \binom{i}{\alpha}$$

$$\frac{\partial u^{\lambda}_{\alpha_s}}{\partial x'_\alpha} + \sum_{i i_s} \begin{Bmatrix} i i_s \\ \lambda \end{Bmatrix} u^i_\alpha u^{i_s}_{\alpha_s} - \sum_r \begin{Bmatrix} \alpha\alpha_s \\ r \end{Bmatrix}' u^{\lambda}_r = \binom{\lambda}{\alpha\alpha_s},$$

so that $\binom{i}{\alpha} = 0$, $\binom{\lambda}{\alpha\alpha_s} = 0$ are the equations by which, in § 6, the derivatives just designated as the unknowns of this system were eliminated from the above equations (G_4'), whereby those transformed into equations (G_5).

If, instead, eliminating these derivatives from equations (G_4') using the above formulas, one obtains on the right side a part U_4, linear in the quantities $\binom{i}{\alpha}, \binom{\lambda}{\alpha\alpha_s}$, which differs from the original right side of (G_4') only in that $\frac{\partial x_i}{\partial x'_\alpha}$ is replaced by $\binom{i}{\alpha}$, and $\frac{\partial u^\lambda_{\alpha_s}}{\partial x'_\alpha}$ is replaced by $\binom{\lambda}{\alpha\alpha_s}$. The remaining terms cancel against the left side, as we assume that equations (G_5) are satisfied. Consequently, equations (G_4') transform into a system of equations

$$U_4 = 0,$$

which is obtained from those by replacing the unknowns with $\binom{i}{\alpha}, \binom{\lambda}{\alpha\alpha_s}$, and suppressing all independent terms.

This system thus also contains a number of equations equal to the number of unknowns $\binom{i}{\alpha}, \binom{\lambda}{\alpha\alpha_s}$, which are independent of each other with respect to those. Consequently, these equations can only be satisfied by generally setting $\binom{i}{\alpha} = 0$, $\binom{\lambda}{\alpha\alpha_s} = 0$ particularly for each i and α

$$u^i_\alpha = \frac{\partial x_i}{\partial x'_\alpha}$$

Thus, if equations (G_4) alone completely determine all unknowns, and the values satisfying them also satisfy equations (G_5) derived from (G_4), then all integrability conditions are satisfied, so that under these assumptions, equations (B) become superfluous.

10.

No new calculation is needed after the above to see that these conclusions still hold if all unknowns are completely determined without contradiction by the systems of equations $(G_p), (G_q), (G_r), \ldots$ and simultaneously satisfy the equations $(G_{p+1}), (G_{q+1}), (G_{r+1}) \ldots$, derived from these according to § 6. Indeed, from each system (G_s) with regard to (G_{s+1}), a system of equations $U_s = 0$ follows, and there must then exist among the set of equations

$$U_p = 0, \quad U_q = 0, \quad U_r = 0, \ldots,$$

a group containing as many equations as there are unknowns, and whose determinant is nonzero.

Among the equations from which the original unknowns x and u are to be determined, the transformation relations (F) themselves are primary; they differ from the other systems of equations to be used, e.g., from equations (G_4), in that the equations derived from them according to § 6 are identically satisfied, since $(i_1 i_2) = \omega_{i_1 i_2}$, $(i i_1 i_2) = 0$ was obtained. Consequently, we obtain the theorem:

From the original differential expression (F), first derive the form G_4 according to sections 5 and 7, and then from this, according to § 6, the series of forms $G_5, G_6, \ldots$. If then, by the transformation relations belonging to the equations

$$F = F', \quad G_4 = G_4', \quad G_5 = G_5', \quad \ldots \quad G_p = G_p'$$

the values of the unknowns x and u are completely determined without contradiction, and the same values of the unknowns also satisfy the transformation relations derived from the next equation

$$G_{p+1} = G'_{p+1},$$

then all integrability conditions necessary for the transformation of F into F' are simultaneously satisfied, and generally

$$u^i_\alpha = \frac{\partial x_i}{\partial x'_\alpha}.$$

Through this theorem, which in all cases where it applies makes the question of integrability conditions superfluous, a new and fruitful field is opened for the applications of algebraic invariant theory.

Indeed, in applying this theorem, the fact that the quantities $\partial x, \underset{1}{\partial x}, \ldots$ represent complete differentials no longer matters, but one must consider $F, G_4, G_5, \ldots$ only as algebraic homogeneous forms with the variable systems $\partial x, \underset{1}{\partial x}, \ldots$, under the assumption that all variable systems are subject to the same linear substitution.

Under this condition, one must now, for purely algebraic reasons, continue the series of successive forms $F, G_4, G_5, \ldots$ until they yield n absolute invariants and as many associated forms Ψ_s, of which the former are independent *with respect to the variables* x, and the latter are independent of each other generally. Then, for the series of forms $F, G_4, G_5, \ldots$ extended by one more term, a complete system of invariants $I, I_1, \ldots$ must be determined, which are independent of each other *with respect to the coefficients of these forms.*

Then the system of all necessary and sufficient conditions for the possibility of the transformation of F into F' consists of the equations

$$I' = r^{\lambda} I, \quad I_1' = r^{\lambda_1} I_1, \quad \ldots,$$

where r is the substitution determinant, and the exponents λ are constant values.

It is appropriate to designate *these invariants $I, I_1, \ldots$ of the sufficiently extended, but not further than necessary, series of forms $F, G_4, G_5, \ldots$ also as a complete system of invariants of the differential expression F.*

Let $U_1, U_2, \ldots, U_n$ be the variables in the original, and $V_1, V_2, \ldots, V_n$ the same in the transformed associated forms, so that for the original substitution

$$\partial x_i = \sum_{\alpha} u_{\alpha}^{i} \partial x_{\alpha}',$$

the transposed substitution

$$V_{\alpha} = \sum_{i} u_{\alpha}^{i} U_i$$

applies. Under this assumption, if the equations between the associated forms are

$$\Psi_s(V_1, V_2, \ldots) = r^{\mu_s} \Psi_s(U_1, U_2, \ldots),$$

for $s = 1, 2, \ldots, n$ and constant values of μ, then, if under Ω an arbitrary function of the original variables is understood, and generally

$$U_i = \frac{\partial \Omega}{\partial x_i}$$

is set, the equations

$$\Psi_s'(\frac{\partial \Omega}{\partial x_1'}, \frac{\partial \Omega}{\partial x_2'}) = r^{\mu_s} \Psi_s(\frac{\partial \Omega}{\partial x_1}, \frac{\partial \Omega}{\partial x_2}, \ldots)$$

follow.

Such expressions containing an arbitrary function Ω

$$\Psi_s\left(\frac{\partial \Omega}{\partial x_1}, \frac{\partial \Omega}{\partial x_2}, \ldots\right)$$

must be designated as associated forms of the differential expression F.

Similarly, the covariants of the system $F, G_4, G_5, \ldots$ yield n equations

$$\Phi_s'(\partial x_1', \partial x_2', \ldots) = r^{v_2} \Phi_s(\partial x_1, \partial x_2, \ldots),$$

and *the differential expressions*

$$\Phi_s(\partial x_1, \partial x_2, \ldots)$$

must be designated as covariants of the differential expression F itself.

11.

To precisely establish the content of the theorem found in the previous section, it must be emphasized that its conditions cannot be satisfied under all circumstances. For example, when it comes to the question of whether two given surfaces can be developed onto each other without stretching, one must investigate an equation $F = F'$ for $n = 2$. If the development of one surface onto the other, i.e., the above equation, is possible, then in those cases where the given surfaces can be continuously displaced within themselves, one will never obtain definite values of x and u from the transformation relations contained in the equations $F = F', G_4 = G_4', G_5 = G_5', \ldots$, no matter how far this series is extended, because in these cases the transformation relations allow continuous changes of the original variables without changes to the new variables.

It is thus easy to specify the conditions for the general case corresponding to the extent a surface can be displaced within itself. The domain of the variables $x_1, x_2, \ldots, x_n$ is called displaceable within itself with respect to an unchanged F, if the substitution to be applied is not completely determined by the result of the transformation of F. It is clear that all conditions for this case to occur consist of identical equations between such invariants and associated forms of F, which, under the assumptions of the previous section, are independent functions of the variables x and u.

In this case, the integrability conditions are not superfluous, because one can satisfy all transformation relations without setting the expressions designated by $\begin{pmatrix} i \\ \alpha \end{pmatrix}, \begin{pmatrix} \lambda \\ \alpha\alpha_s \end{pmatrix}$ in § 9 equal to 0.

For all other cases, the remarkable result holds, which is by no means to be expected a priori from the nature of the matter, that all necessary and sufficient conditions for the possibility of the transformation of F into F' can be represented as equations between invariants, when this term is used to denote the same formal relationships as in algebra, and that in the transformation problem treated here, associated forms and covariants appear in exactly the same sense, albeit from an entirely different source, as in analogous algebraic problems.

12.

As an example of the preceding theory, I treat the case $n = 3$, which offers remarkable simplifications.

If, as in equation (20), for $i_1 i_2$ as well as for $k_1 k_2$, only combinations without repetition of the numbers 123 are taken, e.g., only the groups 23, 31, 12, then the transformation relations (18) of G_4 can be put into the form

$$(\beta_1\beta_2\gamma_1\gamma_2)' = \sum (i_1 i_2 k_1 k_2)(u^{i_1}_{\beta_1} u^{i_2}_{\beta_2} - u^{i_1}_{\beta_2} u^{i_2}_{\beta_1})(u^{k_1}_{\gamma_1} u^{k_2}_{\gamma_2} - u^{k_1}_{\gamma_2} u^{k_2}_{\gamma_1})$$

I now determine four numbers β, γ, i and k by the condition that $\beta\beta_1\beta_2, \gamma\gamma_1\gamma_2, ii_1i_2$ and k, k_1, k_2 become positive permutations of the numbers 123, so that simultaneously, for example,

$$\begin{aligned} &\beta = 1, \quad \beta_1 = 2, \quad \beta_2 = 3, \\ \text{or} \quad &\beta = 2, \quad \beta_1 = 3, \quad \beta_2 = 1, \\ \text{or} \quad &\beta = 3, \quad \beta_1 = 1, \quad \beta_2 = 2 \end{aligned}$$

holds. It is recognized from this that, conversely, both β_1 and β_2 are determined by the value of β.

This established, I denote the subdeterminant of u^i_α arising from a change in r by r^i_α, and set, since $i_1 i_2$ as well as $k_1 k_2$ are completely determined by the values of i and k,

$$(i_1 i_2 k_1 k_2) = A_{ik},$$

so that, according to (14)

$$\text{(a)} \quad \begin{cases} A_{11} = \dfrac{1}{2}\left(2\dfrac{\partial^2\omega_{23}}{\partial x_2 \partial x_3} - \dfrac{\partial^2\omega_{22}}{\partial x_3^2} - \dfrac{\partial^2\omega_{33}}{\partial x_2^2}\right) + \sum\limits_{\alpha\beta} \dfrac{E_{\alpha\beta}}{E}\left(\begin{bmatrix} 23 \\ \alpha \end{bmatrix}\begin{bmatrix} 23 \\ \beta \end{bmatrix} - \begin{bmatrix} 22 \\ \alpha \end{bmatrix}\begin{bmatrix} 33 \\ \beta \end{bmatrix}\right) \\ A_{23} = \dfrac{1}{2}\left(2\dfrac{\partial^2\omega_{23}}{\partial x_1^2} + \dfrac{\partial^2\omega_{11}}{\partial x_2 \partial x_3} - \dfrac{\partial^2\omega_{12}}{\partial x_1 \partial x_3} - \dfrac{\partial^2\omega_{13}}{\partial x_1 \partial x_2}\right) + \sum\limits_{\alpha\beta} \dfrac{E_{\alpha\beta}}{E}\left(\begin{bmatrix} 11 \\ \alpha \end{bmatrix}\begin{bmatrix} 23 \\ \beta \end{bmatrix} - \begin{bmatrix} 12 \\ \alpha \end{bmatrix}\begin{bmatrix} 13 \\ \beta \end{bmatrix}\right) \end{cases}$$

from which the others follow by cyclic permutation of 123.

Then the transformation relations of G_4 take the simple form:

$$A'_{\beta\gamma} = \sum_{ik} A_{ik} r^i_\beta r^k_\gamma. \tag{b}$$

Similarly, the transformation relations belonging to $F = F'$, as evident, for example, from the associated form of F, can be replaced by the following:

$$E'_{\beta\gamma} = \sum_{ik} E_{ik} r^i_\beta r^k_\gamma. \tag{c}$$

It is noted that, due to equations (19),

$$A_{ki} = A_{ik}, \tag{d}$$

and, which will be used later, A_{ik} changes its sign when i_1 and i_2 or k_1 and k_2 are swapped.

Equations (b) and (c) are thus nothing but the transformation relations for the simultaneous transformation of the ternary quadratic forms

$$\Gamma = \sum_{ik} A_{ik} X_i X_k,$$
$$\Phi = \sum_{ik} E_{ik} X_i X_k$$

into the following

$$\Gamma' = \sum_{ik} A'_{ik} \Xi_i \Xi_k,$$
$$\Phi' = \sum_{ik} E'_{ik} \Xi_i \Xi_k$$

by means of the linear substitution

$$X_i = \sum_\beta r^i_\beta \Xi_\beta, \tag{e}$$

whose determinant is

$$R = r^2,$$

and to which the inverse substitution belongs

$$\Xi_\beta = \sum_i \frac{1}{r} u^i_\beta X_i \tag{e'}$$

It is known that this transformation problem is definite, as four simultaneous invariants and three associated forms exist, which are independent with respect to the variables of the latter and the coefficients of Γ, Φ. This yields three absolute invariants; if these are also independent with respect to the variables x_1, x_2, x_3, then, according to the theorem in § 10, for the present investigation, it is only necessary to derive the next form G_5 from F and G_4. Now, according to § 6,

$$(gi_1i_2k_1k_2) = \frac{\partial(i_1i_2k_1k_2)}{\partial x_g} - \sum_\lambda \Bigg[\begin{Bmatrix} gi_1 \\ \lambda \end{Bmatrix} (\lambda i_2k_1k_2) + \begin{Bmatrix} gi_2 \\ \lambda \end{Bmatrix} (i_1\lambda k_1k_2)$$
$$+ \begin{Bmatrix} gk_1 \\ \lambda \end{Bmatrix} (i_1i_2\lambda k_2) + \begin{Bmatrix} gk_2 \\ \lambda \end{Bmatrix} (i_1i_2k_1\lambda) \Bigg].$$

In this sum, for the first term, only the values i_1 and i are to be taken for λ, since $(i_1i_2k_1k_2) = 0$. Proceeding similarly for the other terms and swapping the index with the corresponding one in cases where $\lambda = i$ or $= k$, it follows first

$$(gi_1i_2k_1k_2) = \frac{\partial(i_1i_2k_1k_2)}{\partial x_g} - (i_1i_2k_1k_2)\left[\begin{Bmatrix} gi_1 \\ i_1 \end{Bmatrix} + \begin{Bmatrix} gi_2 \\ i_2 \end{Bmatrix} + \begin{Bmatrix} gk_1 \\ k_1 \end{Bmatrix} + \begin{Bmatrix} gk_2 \\ k_2 \end{Bmatrix}\right]$$
$$+ \begin{Bmatrix} gi_1 \\ i \end{Bmatrix}(i_2ik_1k_2) + \begin{Bmatrix} gi_2 \\ i \end{Bmatrix}(ii_1k_1k_2) + \begin{Bmatrix} gk_1 \\ k \end{Bmatrix}(i_1i_2k_2k) + \begin{Bmatrix} gk_2 \\ k \end{Bmatrix}(i_1i_2kk_1).$$

Now, i_1i_2i, i_2ii_1 and k_1k_2k, k_2kk_1 are, together with ii_1i_2 and kk_1k_2, positive permutations of 123; thus, it follows, if $\begin{Bmatrix} gi \\ i \end{Bmatrix} + \begin{Bmatrix} gk \\ k \end{Bmatrix}$ is added to the bracket of the subtrahend on the right:

$$(gi_1i_2k_1k_2) = \frac{\partial A_{ik}}{\partial x_g} - 2A_{ik}\sum_r \begin{Bmatrix} gr \\ r \end{Bmatrix}$$

$$+ \begin{Bmatrix} gi \\ i \end{Bmatrix} A_{ik} + \begin{Bmatrix} gi_1 \\ i \end{Bmatrix} A_{i_1k} + \begin{Bmatrix} gi_2 \\ i \end{Bmatrix} A_{i_2k} + \begin{Bmatrix} gk \\ k \end{Bmatrix} A_{ik} + \begin{Bmatrix} gk_1 \\ k \end{Bmatrix} A_{ik_1} + \begin{Bmatrix} gk_2 \\ k \end{Bmatrix} A_{ik_2}.$$

I denote this expression, completely determined by the values g, i, k, by A_{gik}, so that

$$(gi_1i_2k_1k_2) = A_{gik},$$

and, according to the easily derived formula,

$$2\sum_r \begin{Bmatrix} gr \\ r \end{Bmatrix} = \frac{1}{E}\frac{\partial E}{\partial x_g}$$

(f) $$A_{gik} = \frac{\partial A_{ik}}{\partial x_{g'}} - \frac{A_{ik}}{E}\frac{\partial E}{\partial x_g} + \sum_\lambda \left[\begin{Bmatrix} g\lambda \\ i \end{Bmatrix} A_{\lambda k} + \begin{Bmatrix} g\lambda \\ k \end{Bmatrix} A_{i\lambda}\right]$$

This expression has the property of remaining unchanged when i is swapped with k, but changing its sign when i_1 is swapped with i_2 or k_1 with k_2. Consequently, the transformation relations belonging to $G_5 = G_5'$

$$A'_{\alpha\beta\gamma} = \sum_{gi_1i_2k_1k_2} A_{gik} u^g_\alpha u^{i_1}_{\beta_1} u^{i_2}_{\beta_2} u^{k_1}_{\gamma_1} u^{k_2}_{\gamma_2}$$

can be brought into the simpler form

(g) $$A'_{\alpha\beta\gamma} = \sum_{gik} A_{gik} u^g_\alpha r^i_\beta r^k_\gamma$$

and now lead to a further remarkable result.

Namely, if for the associated forms of Γ and Φ, $U_1U_2U_3$ are the original variables, and $V_1V_2V_3$ the new variables, so that the corresponding substitution, transposed to (e), is

(h) $$V_\alpha = \sum_g r^g_\alpha U_g,$$

from which, by inversion,

(h') $$U_g = \sum_\alpha \frac{1}{r} u^g_\alpha V_\alpha,$$

follows, then from equation (g), multiplying by $V_\alpha \Xi_\beta \Xi_\gamma$ and summing over α, β, γ, we obtain:

$$\sum_{\alpha\beta\gamma} A'_{\alpha\beta\gamma} V_\alpha \Xi_\beta \Xi_\gamma = r \sum_{gik} A_{gik} U_g X_i X_k.$$

Since $r = R^{\frac{1}{2}}$ is a power of the substitution determinant, and both sums contain both the variables of the original and the associated forms, they must be corresponding simultaneous intermediate forms, and we can thus state the following theorem:

To obtain the necessary and sufficient conditions for the possibility of transforming a ternary quadratic differential expression

$$F = \sum \omega_{ik} \partial x_i \partial x_k,$$

into another

$$F' = \sum \omega'_{ik} \partial x'_i \partial x'_k,$$

and, if these are satisfied, the substitution that accomplishes the required transformation, form, using the coefficients of F, the three algebraic ternary forms

$$\Gamma = \sum A_{ik} X_i X_k,$$
$$\Phi = \sum E_{ik} X_i X_k,$$
$$\Theta = \sum A_{gik} U_g X_i X_k$$

and for F', in the same way, the corresponding forms Γ', Φ', Θ' with the variables Ξ and V in place of X and U.

Then the required conditions are exactly the same as those necessary and sufficient for transforming Γ into Γ', Φ into Φ', and Ω into Ω' by a direct substitution

$$X_i = \sum_\alpha r^i_\alpha \Xi_\alpha,$$

and, under the assumption of the transposed substitution

$$V_\alpha = \sum_i r^i_\alpha U_i,$$

simultaneously Θ and Θ' become corresponding intermediate forms, which satisfy the equation

$$\Theta' = R^{\frac{1}{2}} \Theta$$

However, this theorem holds only under the assumption that the six absolute associated forms and invariants of Γ, Φ are independent functions of the variables $x_1 x_2 x_3 U_1 U_2 U_3$.

If this condition is satisfied, one obtains from the equations between the original and transformed covariants of Γ, Φ, the coefficients $\frac{1}{r} u^i_\beta$ of the inverse substitution, thus, since r is determined by the invariants, u^i_β itself, so that generally

$$u^i_\beta = \frac{\partial x_i}{\partial x'_\beta}$$

holds.

If it is not satisfied, the result follows instead of the above theorem that the domain of the variables x_1, x_2, x_3 can be continuously displaced within itself without changing F.

From § 10, it follows that under the assumptions of the above theorem, all necessary and sufficient conditions for the possibility of transforming F into F' can be represented using the simultaneous invariants of three homogeneous forms F, G_4, G_5 and their transformed forms.

This result, like the above theorem, assumes that in eliminating the substitution coefficients u^i_α or r^i_α, from which the algebraic fundamental forms in question arise, no account is taken of integrability conditions between them.

Now, it is required above that the quantities A_{gik} can be expressed through the coefficients of Γ and Φ, and the quantities A'_{gik} in the same way through the coefficients of Γ' and Φ', such that 1) Θ is a simultaneous intermediate form of Γ and Φ, 2) Θ' is the same simultaneous intermediate form of Γ' and Φ', and 3) $\Theta' = R^{\frac{1}{2}} \Theta$ holds. The exponent of R indicates that these conditions cannot be satisfied if the quantities A_{gik} are rational functions of the quantities A_{ik}, E_{ik}. On the other hand, these conditions can certainly be satisfied if F' is not arbitrarily given but derived from F by a substitution of the for (B), e.g., the identical $x_i = x_i'$. Thus, we have the theorem:

For every ternary quadratic differential expression F, the quantities A_{gik} introduced in equations (f) can be represented as irrational functions of the coefficients of Γ and Φ, such that Ω becomes a simultaneous intermediate form of Γ and Φ, which stands in the relation $\Theta' = R^{\frac{1}{2}}\Theta$ to its transformed form.

Adding to this the equations $\Gamma' = \Gamma, \Phi' = \Phi$, and

$$\sum \pm r_1^1 r_2^2 r_3^3 = R,$$

one obtains 31 transformation relations, from which, although the variables of Θ are subject to different substitutions, the 9 substitution coefficients r_α^i can be eliminated so that all elimination results take the invariant form

$$I' = R^\lambda I.$$

In the main case, where the result of the transformation of Γ and Φ and the substitution to be applied is completely determined, the same necessarily holds for the three functions Γ, Φ, Θ. Then, the number of independent invariants 1) of Γ and Φ alone is 4, 2) of Γ, Φ, Θ together is 22, thus 3) the number of those necessarily containing coefficients of Θ is 18.

According to the terminology used in § 10, the differential expression F in the present case thus has 22 invariants in total and 21 absolute ones, as well as three independent associated forms and as many covariants.

Regarding the differential expressions F excluded by the previous theorem, among which is the square of the line element in three-dimensional space, there exists a paper from Riemann's estate [2], for which Mr. Dedekind[3] has promised the analytical developments omitted there.

3 January 1869.

[2]Translator Note: Bernhard Riemann,"Über die Hypothesen, welche der Geometrie zu Grunde liegen," *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen* 13 (1868): 133–150, originally presented as a habilitation thesis dissertation in 1854.

[3]Translator Note: Richard Dedekind (1831-1916) indeed completed analytical developments for Riemann's posthumous work.