Jackson Classical EM Chapter 2 Notes Part II
AnthonyMakesVideos

2.9 ORTHOGONAL FUNCTIONS AND EXPANSIONS

Consider an interval (a, b) in a variable ξ with a set of real or complex functions $U_n(\xi)$, $n = 1, 2, \ldots,$ square integrable and orthogonal on the interval (a, b). The orthogonality condition on the functions $U_N(\xi)$ is expressed by

$$\int_a^b U_n^\star(\xi) U_m(\xi)\, \mathrm{d}\xi = 0, \quad m \neq n. \tag{1}$$

If $n = m$, the integral is nonzero.
Assume that the functions are normalized so that the integral equals unity, thus the functions are said to be *orthonormal*, and they satisfy

$$\int_a^b U_n^\star(\xi) U_m(\xi)\, \mathrm{d}\xi = \delta_{nm}. \tag{2}$$

An arbitrary function $f(\xi)$ that is square integrable in (a, b) can be expanded in a series of orthonormal functions $U_n(\xi)$. If there are N (finite) terms, consider the "best" coefficients choice a_n to get the "best" representation of $f(\xi)$. Define "best" as minimizing the mean square error M_N:

$$M_N = \int_a^b \left| f(\xi) - \sum_{n=1}^N a_n U_n(\xi) \right|^2 \mathrm{d}\xi \tag{3}$$

with coefficients given by

$$a_n = \int_a^b U_n^\star(\xi) f(\xi)\, \mathrm{d}\xi. \tag{4}$$

$\lim\limits_{N\to\infty} \sum\limits_{n=1}^N a_n U_n(\xi) = f(\xi)$ provided that the set of orthonormal functions is compelte, i.e., there exists a finite N_0 such that for $N > N_0$, $M_N < \epsilon$ such that $\epsilon > 0$. Therefore with the foregoing a_n, $\sum\limits_{n=1}^\infty a_n U_n(\xi) = f(\xi)$ and the sum *"converges in the mean"* to $f(\xi)$.
Thus, $F(\xi)$ with the explicit representation of the coefficients a_n can be written

$$f(\xi) = \int_a^b \left\{ \sum_{n=1}^\infty U_n^\star(\xi') U_n(\xi) \right\} f(\xi')\, \mathrm{d}\xi'. \tag{5}$$

The sum of bilinear terms $U_n^\star(\xi')U_n(\xi)$ must exist only in the neighborhood of $xi' = \xi$. It must be true that

$$\sum_{n=1}^\infty U_n^\star(\xi') U_n(\xi) = \delta(\xi' - \xi) \tag{6}$$

which is the so-called *completeness* or *closure relation*. It is analogous to the orthonormality condition (2) except that the roles of the continuous variable ξ and the discrete index n have been interchanged.

The orthonormal functions for a *Fourier Series* expansion in x on the interval $(-a/2, a/2)$ are

$$\sqrt{\frac{2}{a}}\sin\left(\frac{2\pi mx}{a}\right), \quad \sqrt{\frac{2}{a}}\cos\left(\frac{2\pi mx}{a}\right) \tag{7}$$

where m is a non-negative integer and for $m = 0$ the cosine function is $\frac{1}{\sqrt{a}}$.

The Fourier series expansion of $f(x)$ is customarily written in the form

$$f(x) = \frac{1}{2}A_0 + \sum_{m=1}^{\infty}\left[A_m\cos\left(\frac{2\pi mx}{a}\right) + B_m\sin\left(\frac{2\pi mx}{a}\right)\right] \tag{8}$$

where

$$A_m = \frac{2}{a}\int_{-a/2}^{a/2} f(x)\cos\left(\frac{2\pi mx}{a}\right)\,\mathrm{d}x \tag{9}$$

$$B_m = \frac{2}{a}\int_{-a/2}^{a/2} f(x)\sin\left(\frac{2\pi mx}{a}\right)\,\mathrm{d}x. \tag{10}$$

Suppose that the space is two-dimensional, and the variable ξ ranges over the interval (a, b) while the variable η has the interval (c, d). Denote the orthonormal functions in each dimension as $U_n(\xi)$ and $V_m(\eta)$. Then the expansion of an arbitrary function $f(\xi, \eta)$ is

$$f(\xi,\eta) = \sum_n\sum_m a_{nm}U_n(\xi)V_m(\eta) \tag{11}$$

where

$$a_{nm} = \int_a^b \mathrm{d}\xi\int_c^d \mathrm{d}\eta\; U_n^\star(\xi)V_m^\star(\eta)f(\xi,\eta). \tag{12}$$

If the interval (a, b) becomes infinite, then the set of orthogonal functions $U_n(\xi)$ may become a continuum of functions rather than a denumerable set. In this case, trade the Kroenecker delta for the Dirac delta function.

An important example is the *Fourier integral*; start with the orthonormal set of complex exponentials

$$U_m(x) = \frac{1}{\sqrt{a}}e^{i(2\pi mx)/a}, \quad m = 0, \pm1, \pm2, \ldots \tag{13}$$

on the interval $(-a/2, a/2)$, with the expansion

$$f(x) = \frac{1}{\sqrt{a}}\sum_{m=-\infty}^{\infty} A_m e^{i(2\pi mx)/a} \tag{14}$$

where

$$A_m = \frac{1}{\sqrt{a}}\int_{-a/2}^{a/2} e^{i(2\pi mx)/a}f(x')\,\mathrm{d}x'. \tag{15}$$

Let the interval become infinite ($a \to \infty$) and re-write

$$\frac{2\pi m}{a} \to k, \quad \sum_m \to \int_{-\infty}^{\infty} \mathrm{d}m = \frac{a}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k, \quad A_m \to \sqrt{\frac{2\pi}{a}}A(k). \tag{16}$$

The resulting expansion, equivalent to (14), is the *Fourier integral*,

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)e^{ikx}\,\mathrm{d}k \tag{17}$$

where

$$A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}f(x)\,\mathrm{d}x. \tag{18}$$

The orthogonality condition is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(k-k')x}\,\mathrm{d}x = \delta(k-k') \tag{19}$$

and the completeness relation is

$$\frac{1}{2\pi}\int_{0\infty}^{\infty} e^{ik(x-x')}\,\mathrm{d}k = \delta(x-x'). \tag{20}$$

2.10 SEPARATION OF VARIABLES; LAPLACE EQUATION IN RECTANGULAR COORDINATES

Equations involving the 3D Laplacian operator are known to be separable in 11 different coordinate systems. The Laplace equation in rectangular coordinates is

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0. \tag{21}$$

Assume that the potential can be represented by a product of 3 functions, one for each coordinate:

$$\Phi(x,y,z) = X(x)Y(y)Z(z). \tag{22}$$

Substitute (22) into (21), then divide the result by (2) to yield

$$\frac{1}{X(x)}\frac{\mathrm{d}^2X}{\mathrm{d}x^2} + \frac{1}{Y(y)}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} + \frac{1}{Z(z)}\frac{\mathrm{d}^2Z}{\mathrm{d}z^2} = 0. \tag{23}$$

If (23) holds for arbitrary values of the independent coordinates, each of the three terms must be separately constant:

$$\frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2} = -\alpha^2; \quad \frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2} = -\beta^2; \quad \frac{1}{Z}\frac{\mathrm{d}^2Z}{\mathrm{d}z^2} = \gamma^2 \quad \text{where } \alpha^2+\beta^2=\gamma^2. \tag{24}$$

Arbitrarily choosing α^2 and β^2 to be positive yields

$$X = e^{\pm i\alpha x},\ Y = e^{\pm i\beta y},\ Z = e^{\pm\sqrt{\alpha^2+\beta^2}z}. \tag{25}$$

Thus, (22) $\Rightarrow \Phi = e^{\pm i\alpha x}e^{\pm i\beta y}e^{\pm\sqrt{\alpha^2+\beta^2}z}$ where α and β are still completely arbitrary.
To determine α and β, it is necessary to impose specific boundary conditions on the potential.

Consider a rectangular box with all surfaces kept at zero potential except for the surface $z = c$ at the potential $V(x, y)$, and find the potential everywhere inside the box.

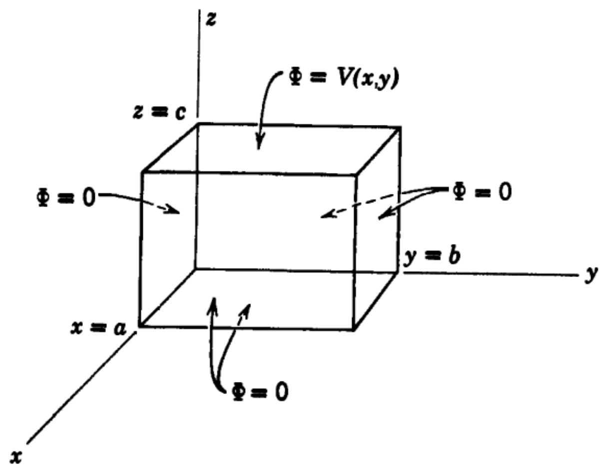

Figure 2.11: Hollow, rectangular box with five sides at zero potential, while the sixth ($z = c$) has the specified potential $\Phi = V(x, y)$.

The requirement that $\Phi = 0$ for $x = 0$, $y = 0$, $z = 0 \Rightarrow$

$$\begin{cases} x = \sin \alpha x \\ y = \sin \beta y \\ z = \sinh(\sqrt{\alpha^2+\beta^2}z) \end{cases} \tag{26}$$

$\Phi = 0$ at $x = a$ and $y = b \Rightarrow \alpha a = n\pi$ and $\beta b = m\pi$. Use

$$\begin{cases} \alpha_n = \dfrac{n\pi}{a} \\ \beta_m = \dfrac{m\pi}{b} \\ \gamma_{nm} = \pi\sqrt{\dfrac{n^2}{a^2}+\dfrac{m^2}{b^2}} \end{cases} \tag{27}$$

to write Φ_{nm}, satisfying all of the boundary conditions except one,

$$\Phi = \sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} z). \tag{28}$$

Expand (28) with initially arbitrary coefficients to obtain the potential

$$\Phi(x,y,z) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} z). \tag{29}$$

Only the boundary condition $\Phi = V(x,y)$ at $z = c$ remains:

$$V(x,y) = \sum_{n,m=1}^{\infty} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} c). \tag{30}$$

This is just a double Fourier series for the function $V(x,y)$ with coefficients A_{nm} given by:

$$A_{nm} = \frac{4}{ab \sinh(\gamma_{nm} c)} \int_0^a \mathrm{d}x \int_0^b \mathrm{d}y\, V(x,y) \sin(\alpha_n x) \sin(\beta_m y). \tag{31}$$

If the rectangular box has potentials different from zero on all six sides, the required solution for the potential inside the box can be obtained by a linear superposition of 6 solutions, one for each side, equivalent to (28) and (31).

The problem of a solution of the Poisson equation (i.e., potential inside the box with a charge distribution inside and prescribed boundary conditions on the surface) requires the construction of the appropriate Green function according to

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{x}') G_D(\mathbf{x},\mathbf{x}')\, \mathrm{d}^3\mathbf{x}' - \frac{1}{4\pi} \oint_S \Phi(\mathbf{x}') \frac{\partial G_D}{\partial n'}\, \mathrm{d}a' \tag{32}$$

such that $G_D(\mathbf{x},\mathbf{x}' = 0$ for $\mathbf{x}'$ on S. (28) and (31) are equivalent to the surface integral in (32).

2.11 A TWO-DIMENSIONAL POTENTIAL PROBLEM; SUMMATION OF A FOURIER SERIES

A two-dimensional problem here refers to those in which the potential can be assumed to be independent of one of the coordinates, say, z. This is usually only an approximation, but may hold true to high accuracy (e.g., a long uniform transmission line).

If the potential is independent of z, the basic solutions of the previous section reduce to the products $e^{\pm i\alpha x} e^{\pm \alpha y}$ where α is any real or complex constant. Imposing boundary conditions on the potential determines what values of α are permitted and the form of the linear superposition of different solutions required.

Consider the problem shown below.

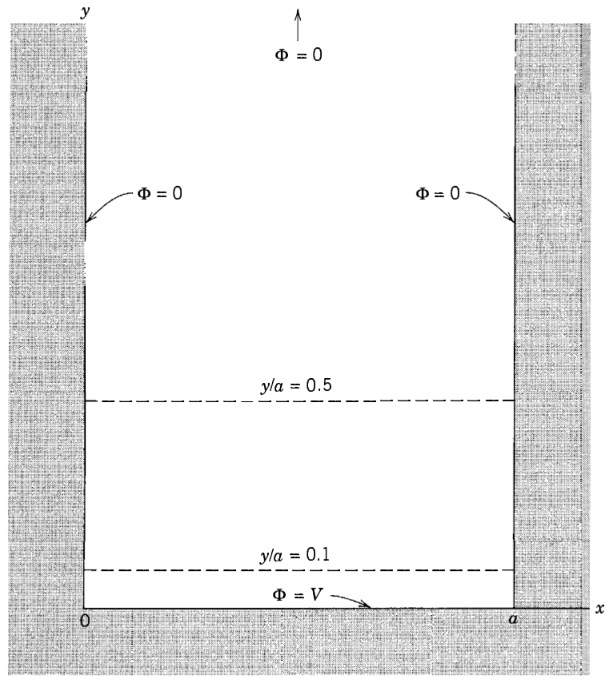

Figure 2.12: Two-dimensional potential problem.

The potential in the region $0 \leq x \leq a$, $y \leq 0$ is desired, subject to the boundary conditions that $\Phi = 0$ at $x = 0$ and $x = a$, while $\Phi = V$ at $y = 0$ for $0 \leq x \leq a$ and $\Phi \to 0$ for large y.

The basic solutions show that that α is real and that, to have the potential vanish at $x = 0$ and $x = a$ for all y and as $y \to \infty$, the proper linear combinations are $e^{-\alpha y} \sin(\alpha x)$ with $\alpha = \frac{n\pi}{a}$. Thus, the linear combination of solutions satisfying the boundary conditions on three of the four boundary surfaces is

$$\Phi(x, y) = \sum_{n=1}^{\infty} A_n \exp\left(\frac{-n\pi y}{a}\right) \sin\left(\frac{n\pi x}{a}\right). \tag{33}$$

The coefficients A_n are determined by the requirement that $\Phi = V$ for $y = 0$, $0 \leq x \leq a$. The Fourier coefficients are

$$A_n = \frac{2}{a} \int_0^a \Phi(x, 0) \sin\left(\frac{n\pi x}{a}\right) \mathrm{d}x. \tag{34}$$

$\Phi(x, 0) = V \Rightarrow$

$$A_n = \frac{4V}{\pi n} \begin{cases} 1 & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases} \tag{35}$$

The potential $\Phi(x, y)$ is therefore determined to be

$$\Phi(x, y) = \frac{4V}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \exp\left(\frac{-n\pi y}{a}\right) \sin\left(\frac{n\pi x}{a}\right). \tag{36}$$

Many terms in the series are necessary for small values of y to give an accurate approximation, but for $y \geq a/\pi$ only the first few terns are appreciable. The potential rapidly approaches its asymptotic form given by the first term,

$$\Phi(x, y) \to \frac{4V}{\pi} \exp\left(\frac{-\pi y}{a}\right) \sin\left(\frac{\pi x}{a}\right). \tag{37}$$

This general behavior is characteristic of all boundary-value problems of this type, independently of whether $\Phi(x, 0$ is a constant, provided the first term in the series is nonvanishing.

The coefficient A_1 (34) will be different, but the smooth behavior in x of the asymptotic solution sets in for $y \geq a$, regardless of the complexities of $\Phi(x, 0)$. This is shown quantitatively in the figure below where the potential is plotted along the two dashed lines, $y/a = 0.1$, 0.5 for the present example. The solid curves are the exact potential and the dotted lines are the first term of (37). Close to the boundary ($y/a = 0.1$) the curves differ appreciably, but for $y/a = 0.5$ the asymptotic form is already an excellent approximation.

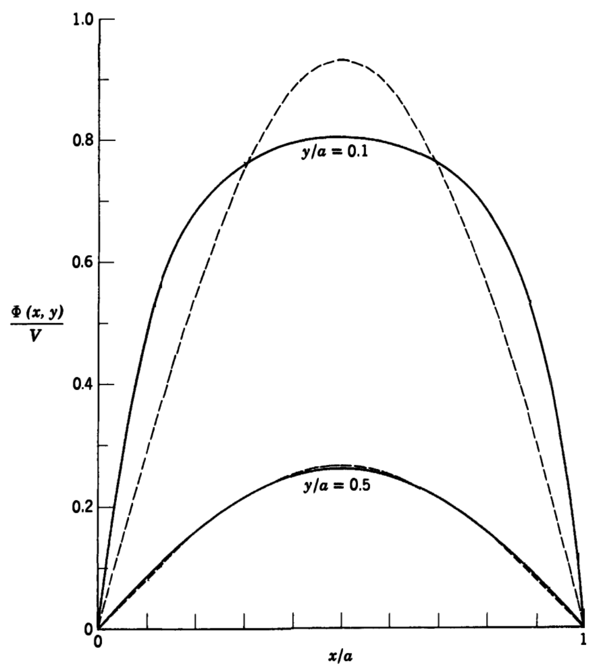

Figure 2.13: Potentials at $y/a = 0.1$, 0.5 (along the dashed lines of Fig. 2.12) as functions of x/a. The solid curves are the exact solution; the dashed curves are the first term in the series solution (36).

There are many Fourier series that can be summed to give an answer in closed form, and the series (36) is one of them.

Observe that $\sin\theta = \text{Im}(e^{i\theta})$, thus (36) can be written as

$$\Phi(x, y) = \frac{4V}{\pi} \text{Im} \sum_{n \text{ odd}} \frac{1}{n} e^{(in\pi/a)(x+iy)}. \tag{38}$$

With the definition

$$Z = e^{(i\pi/a)(x+iy)} \tag{39}$$

this can be rewritten as

$$\Phi(x, y) = \frac{4V}{\pi}\text{Im}\sum_{n\text{ odd}} \frac{Z^n}{n}. \tag{40}$$

Recall the expansion

$$\ln(1 + Z) = Z - \frac{1}{2}Z^2 + \frac{1}{3}Z^3 - \frac{1}{4}Z^4 + \cdots \tag{41}$$

Alternatively, observe that

$$\frac{\mathrm{d}}{\mathrm{d}z}\sum_{n=1}^{\infty} \frac{Z^n}{n} = \sum_{n=0}^{\infty} Z^n = \frac{1}{1 - Z}. \tag{42}$$

Integration of (42) then gives

$$\sum_{n=1}^{\infty} \frac{Z^n}{n} = -\ln(1 - Z). \tag{43}$$

Evidently,

$$\sum_{n\text{ odd}} \frac{Z^n}{n} = \frac{1}{2}\ln\left(\frac{1 + Z}{1 - Z}\right) \tag{44}$$

and

$$\Phi(x, y) = \frac{2V}{\pi}\text{Im}\left[\ln\left(\frac{1 + Z}{1 - Z}\right)\right]. \tag{45}$$

Since the imaginary part of a logarithm is equal to the phase of its argument, consider

$$\frac{1 + Z}{1 - Z} = \frac{(1 + Z)(1 - Z^\star)}{|1 - Z|^2} = \frac{1 - |Z|^2 + 2i\text{Im}Z}{|1 - Z|^2}. \tag{46}$$

The phase of the argument of the logarithm is thus $\tan^{-1}\left[\frac{2\text{Im}Z}{1 - |Z|^2}\right]$. Substituting (39), it is found that the potential becomes

$$\Phi(x, y) = \frac{2V}{\pi}\tan^{-1}\left(\frac{\sin\frac{\pi x}{a}}{\sinh\frac{\pi y}{a}}\right). \tag{47}$$

Note that in connection with (47), the real or imaginary part of an analytic function satisfies the Laplace equation in two dimensions as a result of the Cauchy-Riemann equations.