

## 2.1 METHOD OF IMAGES

The method of images is closely related to the use of Green functions, and concerns itself with the problem of one or more point charges in the presence of boundary surfaces, e.g., conductors either grounded or held at fixed potentials.

The main idea is that it is sometimes possible to infer from the geometry of the situation that a small number of suitably placed charges of appropriate magnitudes, external to the region of interest, can effectively represent the required boundary conditions.

These charges are called “*image charges*”, and the replacement of the actual problem with boundaries by an enlarged region with image charges but no boundaries is called the *method of images*.

The image charges must be external to the volume of interest, since their potentials must be solutions of the Laplace equation inside the volume; the “particular integral” (i.e., the solution of the Poisson equation) is provided by a sum of the potentials of the charges inside the volume.

For a simple example, consider a point charge located in front of an infinite plane conductor at zero potential (see fig. 2.1 below).

This is equivalent to the original problem of the original charge and an equal and opposite charge located at the mirror-image point behind the plane defined by the position of the conductor.

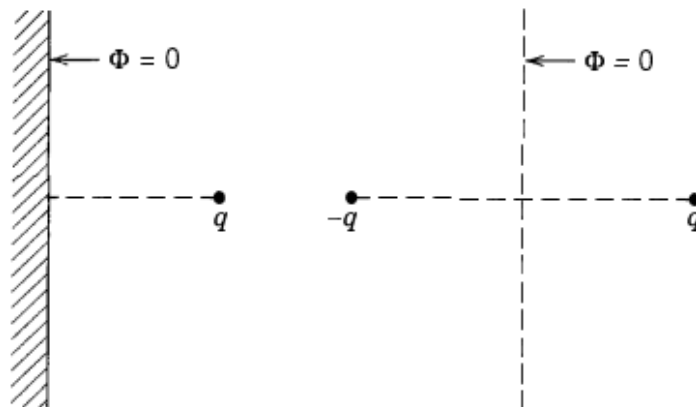


Figure 2.1: Solution by method of images. The original problem is on the left and the equivalent image-problem is on the right.

## 2.2 POINT CHARGE IN THE PRESENCE OF A GROUNDED CONDUCTING SPHERE

To illustrate the method of images, consider the problem depicted in Figure 2.2 below, of a point charge  $q$  located at  $\mathbf{y}$  relative to the origin, around which is centered a grounded conducting sphere of radius  $a$ .

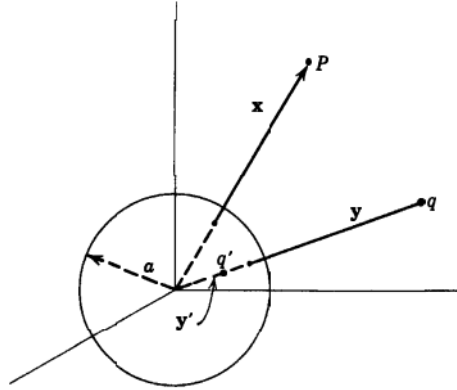


Figure 2.2: Conducting sphere of radius  $a$ , with charge  $q$  and image charge  $q'$ .

A potential  $\Phi(\mathbf{x})$  is sought such that  $\Phi(|\mathbf{x}| = a) = 0$ .

By symmetry it is evident that the image charge  $q'$  (assuming that only one image is needed) will lie on the ray from the origin to the charge  $q$ .

Considering the charge  $q$  *outside* the sphere, the image position  $y'$  will lie inside the sphere.

The potential due to the charges  $q$  and  $q'$  is:

$$\Phi(\mathbf{x}) = \frac{q/4\pi\epsilon_0}{|\mathbf{x} - \mathbf{y}|} + \frac{q'/4\pi\epsilon_0}{|\mathbf{x} - \mathbf{y}'|} \quad (1)$$

Now try to choose  $q'$  and  $|\mathbf{y}'|$  such that this potential vanishes at  $|\mathbf{x}| = a$ .

If  $\mathbf{n}$  is a unit vector in the direction  $\mathbf{x}$ , and  $\mathbf{n}'$  a unit vector in the direction  $\mathbf{y}$ , then

$$\Phi(\mathbf{x}) = \frac{q/4\pi\epsilon_0}{|x\mathbf{n} - y\mathbf{n}'|} + \frac{q'/4\pi\epsilon_0}{|x\mathbf{n} - y'\mathbf{n}'|} \quad (2)$$

Factoring  $x$  out of the first term and  $y'$  out of the second yields the following for the potential at  $x = a$ :

$$\Phi(x = a) = \frac{q/4\pi\epsilon_0}{a|\mathbf{n} - \frac{y}{a}\mathbf{n}'|} + \frac{q'/4\pi\epsilon_0}{y'|\mathbf{n}' - \frac{a}{y'}\mathbf{n}|} \quad (3)$$

From the form of (3) it will be seen that the choices

$$\frac{q}{a} = -\frac{q'}{y'} \quad \frac{y}{a} = \frac{a}{y'}$$

make  $\Phi(x = a) = 0$ , for all possible values of  $\mathbf{n} \cdot \mathbf{n}'$ .

Hence the magnitude and position of the image charge are

$$q' = -\frac{a}{y}q, \quad y' = \frac{a^2}{y}. \quad (4)$$

Note that as the charge  $q$  is brought closer to the sphere, the image charge grows in magnitude and moves out from the center of the sphere.

When  $q$  is just outside the surface of the sphere, the image charge is equal and opposite in magnitude and lies just beneath the surface.

The image charge has thus been found, so return to the original problem of a charge  $q$  outside a grounded conducting sphere and consider various effects.

The actual charge density induced on the surface of the sphere can be calculated from the normal derivative of  $\Phi$  at the surface

$$\sigma = -\epsilon_0 \left. \frac{\partial \Phi}{\partial x} \right|_{x=a} = -\frac{q}{4\pi a^2} \left( \frac{a}{y} \right) \frac{1 - \frac{a^2}{y^2}}{\left( 1 + \frac{a^2}{y^2} - 2\frac{a}{y} \cos \gamma \right)^{3/2}} \quad (5)$$

where  $\gamma$  is the angle between  $\mathbf{x}$  and  $\mathbf{y}$ .

This charge density in units of  $-q/4\pi a^2$  is shown plotted in Figure 2.3 below as a function of  $\gamma$  for two values of  $y/a$ .

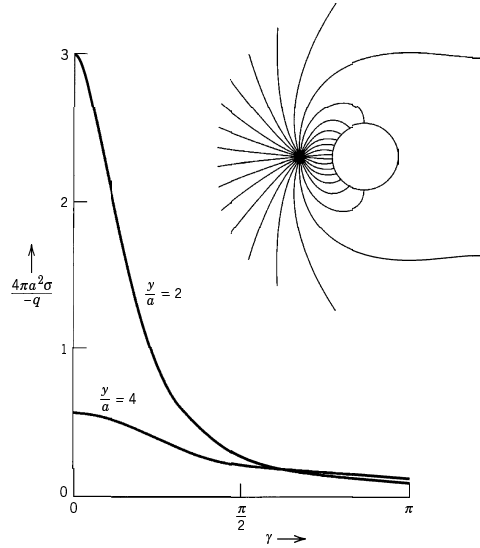


Figure 2.3: Surface-charge density  $\sigma$  induced on the grounded sphere of radius  $a$  as a result of the presence of a point charge  $q$  located a distance  $y$  away from the center of the sphere,  $\sigma$  is plotted in units of  $-q/4\pi a^2$  as a function of the angular position  $\gamma$  away from the radius to the charge for  $y = 2a, 4a$ . Inset shows lines of force for  $y = 2a$ .

The concentration of charge in the direction of the point charge  $q$  is evident, especially for  $y/a = 2$ .

It is easy to show by direct integration that the total induced charge on the sphere is equal to the magnitude of the image charge, as it must be, according to Gauss's law.

The force acting on the charge  $q$  can be calculated in different ways.

The easiest way is to write down immediately the force between the charge  $q$  and the image charge  $q'$ .

The distance between them is  $y - y' = y(1 - a^2/y^2)$ . Hence the attractive force, according to Coulomb's law, is

$$|\mathbf{F}| = \frac{1}{4\pi\epsilon_0} \frac{q^2}{a^2} \left(\frac{a}{y}\right)^3 \left(1 - \frac{a^2}{y^2}\right)^{-2}. \quad (6)$$

For large separations the force is an inverse cube law, but close to the sphere it is proportional to the inverse square of the distance away from the surface of the sphere.

The alternative method for obtaining the force is to calculate the total force acting on the surface of the sphere.

The force on each element of area  $da$  is  $(\sigma^2/2\epsilon_0) da$ , where  $\sigma$  is given by (5), as indicated in Figure 2.4 below.

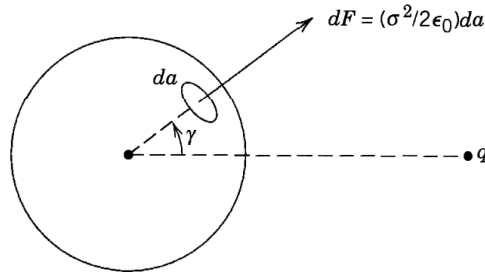


Figure 2.4

But from symmetry it is clear that only the component parallel to the radius vector from the center of the sphere to  $q$  contributes to the total force.

Hence the total force acting on the sphere (equal and opposite to the force acting on  $q$ ) is given by the integral

$$|\mathbf{F}| = \frac{q^2}{32\pi^2\epsilon_0 a^2} \left(\frac{a}{y}\right)^2 \left(1 - \frac{a^2}{y^2}\right)^2 \int \frac{\cos \gamma}{\left(1 + \frac{a^2}{y^2} - \frac{2a}{y} \cos \gamma\right)^3} d\Omega. \quad (7)$$

Integration immediately yields (6).

The foregoing discussion has been based on the understanding that the point charge  $q$  is *outside* the sphere.

Actually, the results apply equally for the charge  $q$  *inside* the sphere.

The only change necessary is in the surface-charge density (5), where the normal derivative out of the conductor is now radially inward, implying a change in sign.

Thereby, all of the preceding formulas may be transcribed for this situation with charge instead inside the sphere, remembering that now  $y \leq a$ .

The angular distributions of surface charge are similar to those of Figure 2.3, but the total induced surface charge is evidently equal to  $-q$ , independent of  $y$ .

## 2.3 POINT CHARGE IN THE PRESENCE OF A CHARGED, INSULATED, CONDUCTING SPHERE

The preceding illustration considered the problem of a point charge  $q$  near a grounded sphere and showed that a surface-charge density was induced on the sphere.

This charge was of total amount  $q' = -aq/y$ , and was distributed over the surface in such a way as to be in equilibrium under all acting forces.

Considering the problem of an insulated conducting sphere with total charge  $Q$  in the presence of a point charge  $q$ , a solution can be built for the potential by linear superposition.

Imagine starting with the grounded conducting sphere with its charge  $q'$  distributed over its surface, then disconnecting the ground wire and adding to the sphere and amount of charge  $(Q - q')$ .

This brings the total charge on the sphere up to  $Q$ .

To find the potential, merely note that the added charge  $(Q - q')$  will distribute itself *uniformly* over the surface, since the electrostatic forces due to the point charge  $q$  are already balanced by the charge  $q'$ .

Hence, the potential due to the added charge  $(Q - q')$  will be the same as if a point charge of that magnitude were at the origin, at least for points outside the sphere.

The potential is the superposition of (1) and the potential of a point charge  $(Q - q')$  at the origin

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \left[ \frac{q}{|\mathbf{x} - \mathbf{y}|} - \frac{aq}{y|\mathbf{x} - \frac{a^2}{y^2}\mathbf{y}|} + \frac{Q + \frac{a}{y}q}{|\mathbf{x}|} \right]. \quad (8)$$

The force acting on the charge  $q$  can be written down directly from Coulomb's law.

It is directed along the radius vector to  $q$  and has the magnitude

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{q}{y^2} \left[ Q - \frac{qa^3(2y^2 - a^2)}{y(y^2 - a^2)^2} \right] \frac{\mathbf{y}}{y} \quad (9)$$

In the limit of  $y \gg a$ , the force reduces to the usual Coulomb's law for two small charged bodies, but close to the sphere the force is modified because of the induced charge distribution on the surface of the sphere.

Figure 2.5 below shows the force as a function of distance for various ratios of  $Q/q$ .

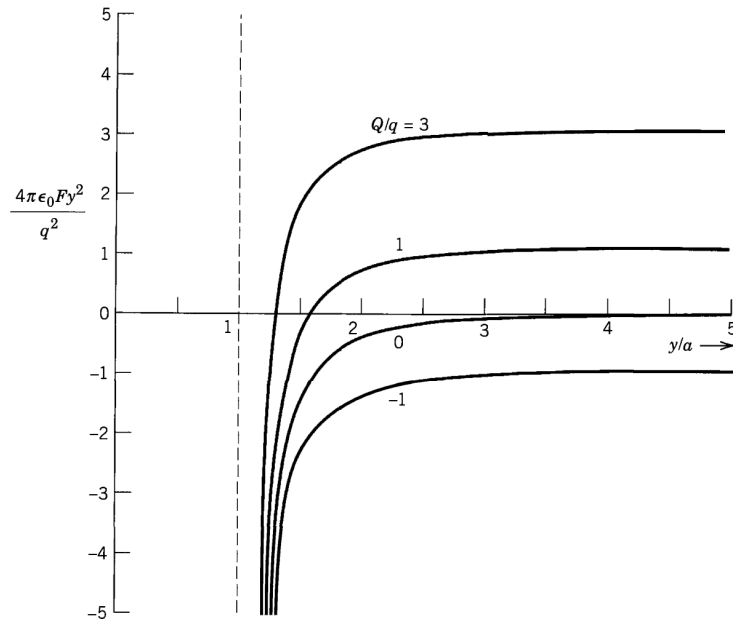


Figure 2.5: The force on a point charge  $q$  due to an insulated, conducting sphere of radius  $a$  carrying a total charge  $Q$ . Positive values mean a repulsion, negative an attraction. The asymptotic dependence of the force has been divided out.  $4\pi\epsilon_0 F y^2 / q^2$  is plotted versus  $y/a$  for  $Q/q = -1, 0, 1, 3$ . Regardless of the value of  $Q$ , the force is always attractive at close distances because of the induced surface charge.

The force is expressed in units of  $q^2/4\pi\epsilon_0 y^2$ ; positive (negative) values correspond to a repulsion (attraction).

If the sphere is charged oppositely to  $q$ , or is uncharged, the force is attractive at very close distances.

Even if the charge  $Q$  is the same sign of  $q$ , however, the force becomes attractive at very close distance.

In the limit of  $Q \gg q$ , the point of zero force (unstable equilibrium point) is very close to the sphere, namely, at  $y \simeq a(1 + \frac{1}{2}\sqrt{q/Q})$ .

Note that the asymptotic value of the force is attained as soon as the charge  $q$  is more than a few radii away from the sphere.

The foregoing example exhibits a general property that explains why an excess of charge on the surface does not immediately leave the surface because of mutual repulsion of the individual charges.

As soon as an element of charge is removed from the surface, the image force tends to attract it back.

If sufficient work is done, of course, charge can be removed from the surface to infinity.

The work function of a metal is in large part just the work done against the attractive image force to remove an electron from the surface.

## 2.4 POINT CHARGE NEAR A CONDUCTING SPHERE AT FIXED POTENTIAL

The potential for a point charge near a conducting sphere held at a fixed potential  $V$  is the same as for the charged sphere, except that the charge  $(Q - q')$  at the center is replaced by a charge  $(4\pi\epsilon_V a)$ .

This can be seen from (8), since at  $|\mathbf{x}| = a$  the first two terms cancel and the last term will be equal to  $V$  as desired.

Thus the potential is

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \left[ \frac{q}{|\mathbf{x} - \mathbf{y}|} - \frac{aq}{y|\mathbf{x} - \frac{a^2}{y^2}\mathbf{y}|} \right] + \frac{Va}{|\mathbf{x}|} \quad (10)$$

The force on the charge  $q$  due to the sphere at fixed potential is

$$\mathbf{F} = \frac{q}{y^2} \left[ Va - \frac{1}{4\pi\epsilon_0} \frac{qay^3}{(y^2 - a^2)^2} \right] \frac{\mathbf{y}}{y} \quad (11)$$

For corresponding values of  $4\pi\epsilon_0 Va/q$  and  $Q/q$  this force is very similar to that of the charged sphere, shown in Figure 2.5, although the approach to the asymptotic value ( $Vaq/y^2$ ) is more gradual.

For  $Va \gg q$ , the unstable equilibrium point has the equivalent location  $y \simeq a(1 + \frac{1}{2}\sqrt{q/4\pi\epsilon_0 Va})$ .

## 2.5 CONDUCTING SPHERE IN A UNIFORM ELECTRIC FIELD BY METHOD OF IMAGES

The method of images is discussed for a conducting sphere in a uniform electric field that has charges outside and inside.

### Two Charges Located Outside the Conducting Sphere

A conducting sphere of a radius  $a$  in a uniform electric field  $E_0$  is considered.

A uniform field can be thought of as being produced by appropriate positive and negative charges at infinity.

E.g., if there are two charges  $\pm Q$ , located at positions  $z = \mp R$ , as shown in Figure 2.6a below, then in a region near the origin whose dimensions are very small compared to  $R$  there is an approximately constant electric field  $E_0 \simeq 2Q/4\pi\epsilon_0 R^2$  parallel to the  $z$  axis.

In the limit as  $R, Q \rightarrow \infty$ , with  $Q/R^2$  constant, this approximation becomes exact.

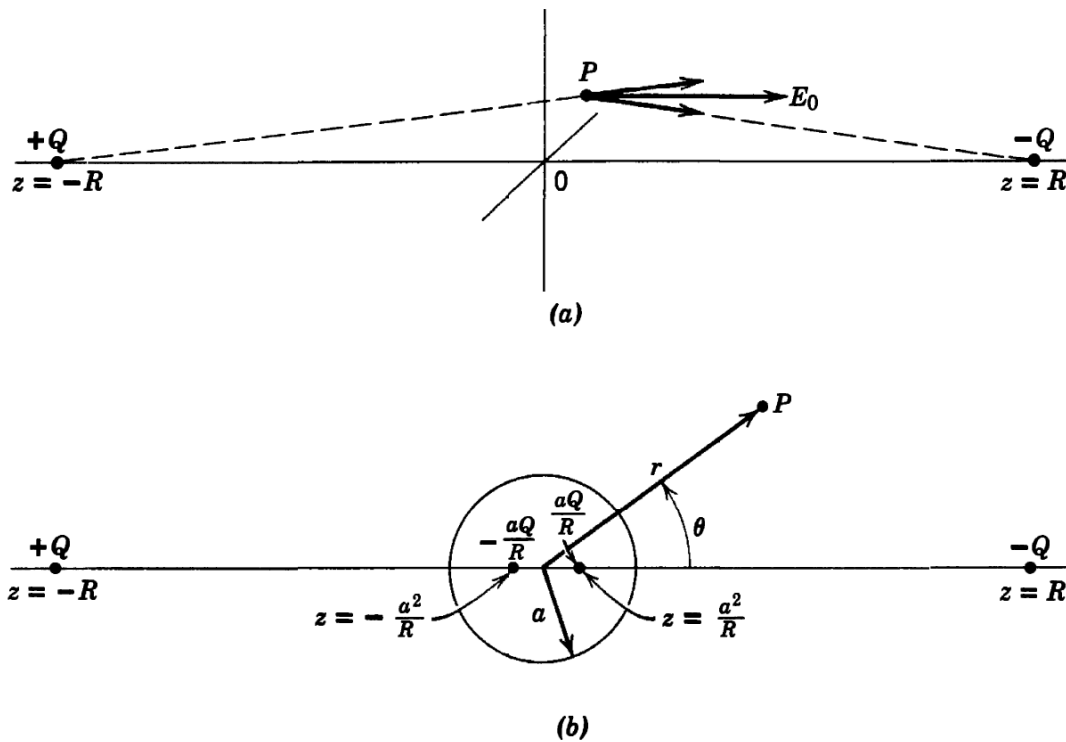


Figure 2.6: Conducting sphere in a uniform electric field by the method of images.

If now a conducting sphere of radius  $a$  is placed at the origin as show in Figure 2.6b above, the potential will be that due to the charges  $\pm Q$  at  $\mp R$  and their images  $\mp QaR$  at  $z = \mp a^2/R$ :

$$\Phi = \frac{Q/4\pi\epsilon_0}{(r^2 + R^2 + 2rR \cos \theta)^{1/2}} - \frac{Q/4\pi\epsilon_0}{(r^2 + R^2 - 2rR \cos \theta)^{1/2}} - \frac{aQ/4\pi\epsilon_0}{R \left( r^2 + \frac{a^4}{R^2} + \frac{2a^2r}{R} \cos \theta \right)^{1/2}} + \frac{aQ/4\pi\epsilon_0}{R \left( r^2 + \frac{a^4}{R^2} - \frac{2a^2r}{R} \cos \theta \right)^{1/2}} \quad (12)$$

where  $\Phi$  has been expressed in terms of the spherical coordinates of the observation point.

In the first two terms  $R$  is much larger than  $r$  by assumption.

Hence the radicals can be expanded after factoring out  $R^2$ .

Similarly, in the third and fourth terms,  $r^2$  can be factored out, and the result can then be expanded. This result is

$$\Phi = \frac{1}{4\pi\epsilon_0} \left[ -\frac{2Q}{R^2} r \cos \theta + \frac{2Q}{R^2} \frac{a^3}{r^2} \cos \theta \right] + \dots \quad (13)$$

where the omitted terms vanish in the limit  $R \rightarrow \infty$ .

In that limit  $2Q/4\pi\epsilon_0 R^2$  becomes the applied uniform field, so that the potential is

$$\Phi = -E_0 \left( r - \frac{a^3}{r^2} \right) \cos \theta \quad (14)$$

The first term ( $-E_0 z$ ) is, of course, just the potential of a uniform field  $E_0$ , which could have been written down directly instead of the first two terms in (12).

The second term is the potential due to the induced surface-charge density or, equivalently, the image charges.

Note that the image charges form a dipole of strength  $D = Qa/R \times 2a^2 R = 4\pi\epsilon_0 E_0 a^3$ .

The induced surface-charge density is

$$\sigma = -\epsilon_0 \frac{\partial \Phi}{\partial r} \Big|_{r=a} = 3\epsilon_0 E_0 \cos \theta. \quad (15)$$

Note that the surface integral of this charge density vanishes, so that there is no difference between a grounded and an insulated sphere.

## A Point Charge Inside the Conducting Sphere

The following subsection discusses the method of images for a point charge  $q$  inside a hollow, grounded, conducting sphere of inner radius  $a$  as shown in Figure 2.7 below.

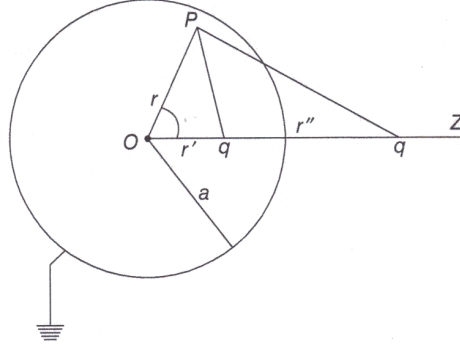


Figure 2.7

If the distance between the charge  $q$  and the origin is  $r$ , then the potential inside the sphere is

$$\Phi(\vec{r}) = \frac{q}{|r - r'|} + \frac{q'}{|r - r''|} \quad (16)$$

where  $q' = \frac{-qa}{r'}$  (see (4)).

By substitution, (16) becomes

$$\Phi(\mathbf{r}) = q \left\{ \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \theta}} - \frac{a/r'}{\sqrt{r^2 + \left(\frac{a^2}{r'}\right)^2 - 2r\frac{a^2}{r'} \cos \theta}} \right\}. \quad (17)$$

This is true both inside and outside of the grounded conducting sphere.

The induced surface-charge density  $\sigma$  inside the conducting sphere is given by

$$\sigma = \frac{-1}{4\pi} \frac{\partial \Phi}{\partial n} = \frac{-1}{4\pi} \frac{\partial \Phi}{\partial r} \Big|_{r=a} \quad (18)$$

$$\sigma = \frac{-q}{4\pi} \left( \frac{a}{r'^3} \right) \frac{(1 - r'^2/a^2)}{(1 + a^2/r'^2 - 2a/r' \cos \theta)^{3/2}}. \quad (19)$$

The total charge  $Q$  is then

$$Q = \int_0^\pi 2\pi a^2 \sigma \, d\theta = \frac{2\pi a^2 q}{4\pi} \left( \frac{r'^2}{a} - a \right) \int_0^\pi \frac{\sin \theta \, d\theta}{(r'^2 + a^2 - 2ar' \cos \theta)^{3/2}} \quad (20)$$

$$= \frac{q(r'^2 - a^2)}{2r'} \left\{ \frac{1}{(r'^2 + a^2 - 2ar')^{1/2}} - \frac{1}{(r'^2 + a^2 + 2ar')^{1/2}} \right\} \quad (21)$$

$$= \frac{qa}{r'} = -q'. \quad (22)$$

Now, the force acting on  $q$  when inside the grounded conducting sphere depends only on the field due to the sphere at the location of  $q$ .

This field is the same as that produced by the image charge; hence the force on  $q$  is the same as that due to the image charge

$$|\mathbf{F}| = \left| \frac{qq'}{(r'' - r')^2} \right| = \frac{q^2 ar'}{(a^2 - r'^2)^2} \quad (23)$$

since  $r'' = \frac{a^2}{r'}$  (see (4)).

The force on  $q$  is radially outwards from the center of the sphere.

The total force on the sphere must be equal and opposite to that on the charge since the total force must be zero.

### Worked Problem 2.1

The method of images will be used to consider a point charge  $q$  inside a hollow, grounded, conducting sphere of inner radius  $a$ , whereby the potential will be found if the sphere is kept at a fixed potential  $V$ .

Also, the case will be considered where the sphere has a total charge  $Q$  on its inner and outer surfaces to find the induced surface charge density, and the direction and magnitude of force acting on charge  $q$ .

### Solution

If the sphere, instead of being grounded, is held at a fixed potential  $V$  or is given a total charge  $Q$ , there is no change in the fields inside the sphere as discussed in **§2.5 Two Charges Located Outside the Conducting Sphere** ( $E_0 = 2Q/4\pi\epsilon_0 R^2$ ).

Also, the charge density on the inner surface remains the same as given in (19) and the force acting is given by (22).

The charge on the external surface is no longer; rather, there is a field

$$\Phi = \frac{V}{r} \quad \text{for } r \gg a$$

and the internal potential is increased by  $V$  over the value given in (17).

The charge on the exterior of the surface of the sphere is  $Va$ , the charge on the interior surface is  $-q$  as before, and the total charge on the sphere is

$$Q = Va - q \Rightarrow V = \frac{Q + q}{a}$$

The charge density on the inner surface is the same as that found in (18).

The charge density on the outer surface is  $V/(4\pi a)$ .

In computing force

$$F_S = 2\pi\sigma^2$$

one must split into two parts and get the proper direction for each.

Note that conductors must have a thickness, but surface does not need to have it.

Thus, one has

$$F_S = \frac{1}{2}\sigma(E_2 + E_1)$$

## 2.6 GREEN FUNCTION FOR THE SPHERE; GENERAL SOLUTION FOR THE POTENTIAL

The previous sections discussed the problem of a conducting sphere in the presence of a point charge using the method of images.

The potential due to a unit source and its image (or images), chosen to satisfy homogeneous boundary conditions, has a general representation given by the Green function appropriate for Dirichlet or Neumann boundary conditions.

In  $G(\mathbf{x}, \mathbf{x}')$  the variable  $\mathbf{x}'$  refers to the location  $P'$  of the unit source, while the variable  $\mathbf{x}$  is the point  $P$  at which the potential is being evaluated.

These coordinates and the sphere are shown in Figure 2.8 below.

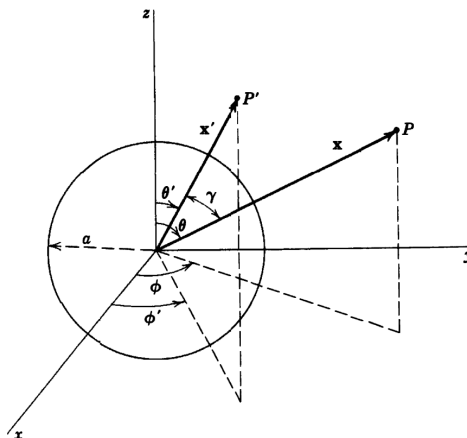


Figure 2.8

For Dirichlet boundary conditions on the sphere of radius  $a$  the Green function defined via

$$\nabla'^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}') \quad \text{where} \quad G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} + F(\mathbf{x}, \mathbf{x}') \quad (24)$$

with the function  $F$  satisfying Laplace's equation inside the volume  $V$ , i.e.,  $\nabla^2 F = 0$ , for a unit source and its image given by (1) with  $q \rightarrow 4\pi\epsilon_0$  and relations (4).

Transforming variables approximately, one obtains the Green function

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} - \frac{a}{x' \left| \mathbf{x} - \frac{a^2}{x'^2} \mathbf{x}' \right|} \quad (25)$$

In terms of spherical coordinates this can be written

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{(x^2 + x'^2 - 2xx' \cos \gamma)^{1/2}} - \frac{1}{\left( \frac{x^2 x'^2}{a^2} + a^2 - 2xx' \cos \gamma \right)^{1/2}} \quad (26)$$

where  $\gamma$  is the angle between  $\mathbf{x}$  and  $\mathbf{x}'$ .

The symmetry in the variables  $\mathbf{x}$  and  $\mathbf{x}'$  is obvious in the form (26), as is the condition that  $G = 0$  if either  $\mathbf{x}$  or  $\mathbf{x}'$  is on the surface of the sphere.

For a solution of Poisson's equation with *Dirichlet boundary conditions* (i.e., the potential on a closed surface is specified), namely

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{x}') G_D(\mathbf{x}, \mathbf{x}') d^3x' - \frac{1}{4\pi} \oint_S \Phi(\mathbf{x}') \frac{\partial G_D}{\partial n'} da', \quad (27)$$

we need not only  $G$  but also  $\partial G / \partial n'$ .

Recall that  $\mathbf{n}'$  is the unit normal outward from the volume of interest (i.e., inward along  $\mathbf{x}'$  toward the origin), thus

$$\left. \frac{\partial G}{\partial n'} \right|_{x'=a} = - \frac{(x^2 - a^2)}{a(x^2 + a^2 - 2ax \cos \gamma)^{3/2}} \quad (28)$$

Note that this is essentially the induced surface-charge density (5).

Hence the solution of Laplace's equation *outside* a sphere with the potential specified on its surface is, according to (27),

$$\Phi(\mathbf{x}) = \frac{1}{4\pi} \int \Phi(a, \theta', \phi') \frac{a(x^2 - a^2)}{(x^2 + a^2 - 2ax \cos \gamma)^{3/2}} d\Omega' \quad (29)$$

where  $d\Omega'$  is the element of solid angle at the point  $(a, \theta', \phi')$  and  $\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi')$ .

For the *interior problem*, the normal derivative is radially outward, so that the sign for  $\partial G/\partial n'$  is opposite to (28).

This is equivalent to replacing the factor  $(x^2 - a^2)$  by  $(a^2 - x^2)$  in (29).

For a problem with a charge distribution, one must add to (29) the appropriate integral in (27), with the Green function (26).

## 2.7 CONDUCTING SPHERE WITH HEMISPHERES AT DIFFERENT POTENTIALS

Consider the conducting sphere of radius  $a$  made up of two hemispherical shells separated by a small insulating ring as an example of the solution (29) for the potential outside a sphere with prescribed values of potential on its surface.

The hemispheres are kept at different potentials.

It will suffice to consider the potentials as  $\pm V$ , since arbitrary potentials can be handled by superposition of the solution for a sphere at fixed potential over its whole surface.

The insulating ring lies in the  $z = 0$  plane, as shown in Figure 2.9 below, with the upper (lower) hemisphere at potential  $+V$  ( $-V$ ).

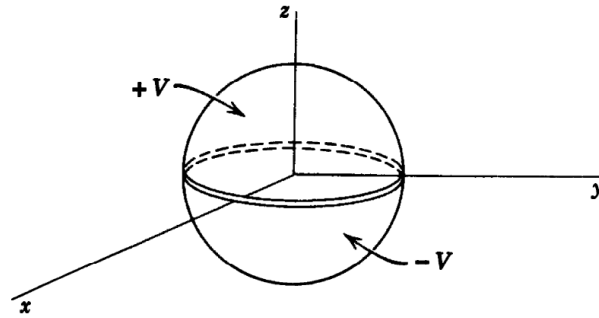


Figure 2.9

From (29) the solution for  $\Phi(x, \theta, \phi)$  is given by the integral

$$\Phi(x, \theta, \phi) = \frac{V}{4\pi} \int_0^{2\pi} d\phi' \left\{ \int_0^1 d(\cos \theta') - \int_{-1}^0 d(\cos \theta') \right\} \frac{a(x^2 - a^2)}{(a^2 + x^2 - 2ax \cos \gamma)^{3/2}} \quad (30)$$

By a suitable change of variables in the second integral of  $(\theta' \rightarrow \pi - \theta', \phi' \rightarrow \phi' + \pi)$ , (30) can be cast in the form

$$\Phi(x, \theta, \phi) = \frac{Va(x^2 - a^2)}{4\pi} \int_0^{2\pi} d\phi' \int_0^1 d(\cos \theta') [(a^2 + x^2 - 2ax \cos \gamma)^{-3/2} - (a^2 + x^2 + 2ax \cos \gamma)^{-3/2}] \quad (31)$$

Because of the complicated dependence of  $\cos \gamma$  on the angles  $(\theta', \phi')$  and  $(\theta, \phi)$ , equation (31) cannot be in general integrated in closed form.

As a special case, consider the potential on the  $z$  axis.

Then  $\cos \gamma = \cos \theta'$ , since  $\theta = 0$ .

Integration yields that the potential is

$$\Phi(z) = V \left[ 1 - \frac{(z^2 - a^2)}{z\sqrt{z^2 + a^2}} \right]. \quad (32)$$

At  $z = a$ , this reduces to  $\Phi = V$  as required, while at large distances it goes asymptotically as  $\Phi \simeq 3Va^2/2z^2$ .

In the absence of a closed form expression for the integrals in (31), the denominator can be expanded in power series and integrates term-by-term.

Factoring out  $(a^2 + x^2)$  from each denominator yields

$$\Phi(x, \theta, \phi) = \frac{Va(x^2 - a^2)}{4\pi(x^2 + a^2)^{3/2}} \int_0^{2\pi} d\phi' \int_0^1 d(\cos \theta') [(1 - 2\alpha \cos \gamma)^{-3/2} - (1 + 2\alpha \cos \gamma)^{-3/2}] \quad (33)$$

where  $\alpha = ax/(a^2 + x^2)$ .

Observe that in the expansion of the radicals only odd powers of  $\alpha \cos \gamma$  will appear:

$$[(1 - 2\alpha \cos \gamma)^{-3/2} - (1 + 2\alpha \cos \gamma)^{-3/2}] = 6\alpha \cos \gamma + 35\alpha^3 \cos^3 \gamma + \dots \quad (34)$$

It is now necessary to integrate odd powers of  $\cos \gamma$  over  $d\phi' d(\cos \theta')$ :

$$\left. \begin{aligned} \int_0^{2\pi} d\phi' \int_0^1 d(\cos \theta') \cos \gamma &= \pi \cos \theta \\ \int_0^{2\pi} d\phi' \int_0^1 d(\cos \theta') \cos^3 \gamma &= \frac{\pi}{4} \cos \theta (3 - \cos^2 \theta) \end{aligned} \right\} \quad (35)$$

If (34) and (35) are inserted into (33), the potential becomes

$$\Phi(x, \theta, \phi) = \frac{3Va^2}{2x^2} \left( \frac{x^3(x^2 - a^2)}{(x^2 + a^2)^{5/2}} \right) \cos \theta \left[ 1 + \frac{35}{24} \frac{a^2 x^2}{(a^2 + x^2)^2} (3 - \cos^2 \theta) + \dots \right] \quad (36)$$

Note that only odd powers of  $\cos \theta$  appear, as required by the symmetry of the problem.

If the expansion parameter is  $(a^2/x^2)$ , rather than  $\alpha^2$ , the series takes the form:

$$\Phi(x, \theta, \phi) = \frac{3Va^2}{2x^2} \left[ \cos \theta - \frac{7a^2}{12x^2} \left( \frac{5}{2} \cos^3 \theta - \frac{3}{2} \cos \theta \right) + \dots \right]. \quad (37)$$

For large values of  $x/a$  this expansion converges rapidly, and so is a useful representation for the potential.

Even for  $x/a = 5$ , the second term in the series is only of the order of 2%.

It can be verified that, for  $\cos \theta = 1$ , the expression (37) agrees with the expansion of (32) for the potential on the axis.

The particular choice of angular factors in (37) is dictated by the definitions of the Legendre polynomials.

The two factors are, in fact,  $P_1(\cos \theta)$  and  $(P_3(\cos \theta))$ , and the expansion of the potential is one in Legendre polynomials of odd order.

### Worked Problem 2.2

Consider a potential problem in the half-space defined by  $z \geq 0$ , with Dirichlet boundary conditions on the plane  $z = 0$  (and at infinity).

(a) Write down the appropriate Green function  $G(\mathbf{x}, \mathbf{x}')$ .

The form of the Green's function given in chapter 1 of the Jackson book is

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} + F(\mathbf{x}, \mathbf{x}')$$

Dirichlet boundary conditions specify at the surface  $G(\mathbf{x}, m\mathbf{x}') = 0$ , and likewise  $G(\mathbf{x}, m\mathbf{x}') = 0$  for  $z < 0$ .

It is given that at  $z = 0$  and  $z = \infty$ ,  $\Phi = \text{constant} = V$ .

Use cylindrical coordinates and drop  $F(\mathbf{x}, \mathbf{x}')$ .

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos \phi + (z - z')^2}} + \frac{1}{\sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos \phi + (z - z)^2}}$$

The second term is the contribution from the image

$$\left. \frac{\partial G}{\partial z'} \right|_{z'=0} = \frac{2z}{[\rho^2 + \rho'^2 - 2\rho\rho' \cos \phi + z^2]^{3/2}}$$

- (b) If the potential on the plane  $z = 0$  is specified to be  $\Phi = V$  inside a circle of radius  $a$  centered at the origin, and  $\Phi = 0$  outside the circle, find an integral expression for the potential at the point  $P$  specified in terms of cylindrical coordinates  $(\rho, \phi, z)$ .

The integral expression for the potential is

$$\Phi(\rho, \phi, z) = -\frac{1}{4\pi} \int_0^{2\pi} \frac{2zV}{[\rho^2 + \rho'^2 - 2\rho\rho' \cos \phi + z^2]^{3/2}} \rho' d\rho' d\phi$$

- (c) Show that, along the axis of the circle ( $\rho = 0$ ), the potential is given by

$$\Phi = V \left( 1 - \frac{z}{\sqrt{a^2 + z^2}} \right).$$

For potential along the axis of a circle, consider the  $z$ -axis ( $\rho = 0$ ), as shown in Figure 2.10 below.

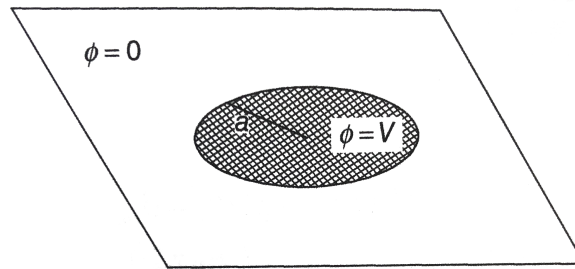


Figure 2.10

$$\begin{aligned} \Phi(0, \phi, z) &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^a \frac{zV}{[\rho'^2 + z^2]^{3/2}} \rho' d\rho' d\phi \\ &= V \left[ 1 - \frac{z}{\sqrt{a^2 + z^2}} \right]. \end{aligned}$$

(d) Show that at large distances ( $\rho^2 + z^2 \gg a^2$ ) the potential can be expanded in a power series in  $(\rho^2 + z^2)^{-1}$ , and that the leading terms are

$$\Phi = \frac{Va^2}{2} \frac{z}{(\rho^2 + z^2)^{3/2}} \left[ 1 - \frac{3a^2}{4(\rho^2 + z^2)} + \frac{5(3\rho^2 a^2 + a^4)}{8(\rho^2 + z^2)^2} + \dots \right]$$

Verify that the results of parts c and d are consistent with each other in their common range of validity.

At a large distance ( $\rho'^2 + z^2 \gg a^2$ )

$$\Phi(\rho, \phi, z) = \frac{Vz}{2\pi} \int_0^a \frac{\rho'}{[\rho^2 + \rho'^2 - 2\rho\rho' \cos \phi + z^2]^{3/2}} d\rho' d\phi$$

since  $(\rho'^2 + z^2) \gg a^2$ , the denominator can be expanded as

$$\begin{aligned} \frac{1}{[\rho^2 + \rho'^2 - 2\rho \cdot \rho' \cos \phi + z^2]^{3/2}} &= \frac{1}{(\rho^2 + z^2)^{3/2}} \left\{ 1 - \left(\frac{3}{2}\right) \frac{\rho'^2 - 2\rho\rho' \cos \phi}{\rho^2 + z^2} \right. \\ &\quad \left. + \frac{15(\rho'^2 - 2\rho \cdot \rho' \cos \phi)^2}{8(\rho^2 + z^2)^2} + \dots \right\} \\ \Phi(\rho, \phi, z) &= \frac{Va^2}{2} \frac{z}{(\rho^2 + z^2)^{3/2}} \left\{ 1 - \frac{3a^2}{4(\rho^2 + z^2)} + \frac{5(3\rho^2 a^2 + a^4)}{8(\rho^2 + z^2)^2} + \dots \right\} \end{aligned}$$

The result (c) is modified by setting  $z^2 \gg a^2$ , thus the expansion is

$$\Phi = V \left[ 1 - \frac{z}{z(1 + a^2/z^2)^{1/2}} \right] = V \left[ \frac{a^2}{2z^2} - \frac{3}{8} \frac{a^4}{z^4} + \dots \right]$$

While the result of the expansion in part (d) can be written along the z axis ( $\rho = 0$ )

$$\Phi = \frac{Va^2}{2az^2} \left[ 1 - \frac{3a^2}{4z^2} + \dots \right]$$

These two expressions agree.