Evans PDE Chapter 2 Notes
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2.2 LAPLACE'S EQUATION

Laplace's equation

$$\Delta u = 0 \tag{1}$$

Poisson's equation

$$-\Delta u = f \tag{2}$$

Laplacian of u

$$\Delta u := \sum_{i=1}^{n} u_{x_i x_i} \tag{3}$$

DEFINITION. A C^2 function u satisfying (1) is called a *harmonic* function.

In a typical interpretation u denotes the density of some quantity (e.g., a chemical concentration) in equilibrium. Then if V is any smooth subregion within U, the net flux of u through ∂V is zero:

$$\int_{\partial V} \mathbf{F} \cdot \nu \, \mathrm{d}S = 0,$$

$\mathbf{F}$ denoting the flux density and ν the unit outer normal field.

Recall the Gauss-Green Theorem.

THEOREM 1 (Gauss-Green Theorem).

(i) *Suppose* $u \in C^1(\bar{U})$. *Then*

$$\int_U u_{x_i} \, \mathrm{d}x = \int_{\partial U} u \nu^i \, \mathrm{d}S \quad (i = 1, \dots, n). \tag{4}$$

(ii) *Thus one has the "Divergence Theorem":*

$$\int_U \operatorname{div} \mathbf{u} \, \mathrm{d}x = \int_{\partial U} \mathbf{u} \cdot \nu \, \mathrm{d}S \tag{5}$$

for each vector field $\mathbf{u} \in C^1(\bar{U}; \mathbb{R}^n)$.

In view of the Gauss-Green Theorem, one has

$$\int_V \text{div}\,\mathbf{F}\,\mathrm{d}x = \int_{\partial V} \mathbf{F}\cdot\nu\,\mathrm{d}S = 0,$$

and so

$$\text{div}\mathbf{F} = 0 \quad \text{in } U, \tag{6}$$

since V was arbitrary. In many instances, it is physically reasonable to assume the flux $\mathbf{F}$ is proportional to the gradient Du but points in the opposite direction (since the flow is from regions of higher to lower concentration). Thus

$$\mathbf{F} = -aDu \quad (a > 0). \tag{7}$$

Substituting (7) into 6, one obtains Laplace's equation

$$\text{div}(Du) = \Delta u = 0.$$

If u denotes the

$$\begin{cases} \text{chemical concentration} \\ \text{temperature} \\ \text{electrostatic potential,} \end{cases}$$

equation (7) is

$$\begin{cases} \text{Fick's law of diffusion} \\ \text{Fourier's law of heat conduction} \\ \text{Ohm's law of electrical conduction.} \end{cases}$$

Laplace's equation also arises in the study of analytic functions and the probabilistic investigation of Brownian motion.

2.2.1 Fundamental solution.

a. **Derivation of fundamental solution.** Since Laplace's equation is invariant under rotations, it consequently seems advisable to search first for *radial* solutions, i.e., functions of $r = |x|$.

Attempt to fund a solution u of Laplace's equation (1) in $U = \mathbb{R}^n$, having the form

$$u(x) = v(r),$$

where $r = |x| = (x_1^2 + \cdots + x_n^2)^{1/2}$ and v is selected (if possible) so that $\Delta u = 0$ holds. Note for $i = 1, \ldots, n$ that

$$\frac{\partial r}{\partial x_i} = \frac{1}{2}(x_1^2 + \ldots + x_n^2)^{-1/2} 2x_i = \frac{x_i}{r} \quad (x \neq 0).$$

$$\Rightarrow u_{x_i} = v'(r)\frac{x_i}{r},\ u_{x_ix_i} = v''(r)\frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right) \quad \text{for } i = 1, \ldots, n$$

$$\Rightarrow \Delta u = 0 = v''(r) + \frac{n-1}{r}v'(r).$$

Hence

$$\Delta u = 0 \Leftrightarrow v'' + \frac{n-1}{r} v' = 0.$$

If $v' \neq 0 \;\Rightarrow$

$$\log(|v'|)' = \frac{v''}{v'} = \frac{1-n}{r} \Rightarrow v'(r) = \frac{a}{r^{n-1}} \text{ for some const. } a.$$

Consequently if $r > 0 \;\Rightarrow$

$$v(r) = \begin{cases} b \log r + c & (n = 2) \\ \dfrac{b}{r^{n-2}} + c & (n \geq 3), \end{cases}$$

where b and c are constants.

These considerations motivate the following

DEFINITION. The function

(8) $$\Phi(x) := \begin{cases} -\frac{1}{2\pi} \log |x| & (n = 2) \\ \dfrac{1}{n(n-2)\alpha(n)} \dfrac{1}{|x|^{n-2}} & (n \geq 3), \end{cases}$$

defined for $x \in \mathbb{R}^n$, $x \neq 0$, is the *fundamental solution of Laplace's equation.*

Note that $\alpha(n)$ denotes the volume of the unit ball in $\mathbb{R}^n$.

Sometimes notation will be slightly abused and it will be written $\Phi(x) = \Phi(|x|)$ to emphasize that the fundamental solution is radial.

Observe that one has the following estimates

(9) $$|D\Phi(x)| \leq \frac{C}{|x|^{n-1}}, \; |D^2\Phi(x)| \leq \frac{C}{|x|^n} \quad (x \neq 0)$$

for some constant $C > 0$.

b. **Poisson's equation.** "*Harmonic*" refers to a function that is a solution of Laplace's equation. By construction the function $x \mapsto \Phi(x)$ is harmonic for $x \neq 0$. If the origin is shifted to a new point y, the PDE (1) is unchanged; and so $x \mapsto \Phi(x-y)$ is also harmonic as a function of x, $x \neq y$. Now take $f : \mathbb{R}^n \to \mathbb{R}$ and note that the mapping $x \mapsto \Phi(x-y)f(y)$ where $x \neq y$ is harmonic for each point $y \in \mathbb{R}^n$, and thus so is the sum of finitely many such expressions built for different points y.

This line of reasoning might suggest that the convolution

$$
\begin{aligned}
u(x) &= \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, \mathrm{d}y \\
&= \begin{cases} -\dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}^n} \log\left(|x-y|\right) f(y)\, \mathrm{d}y & (n=2) \\ \dfrac{1}{n(n-2)\alpha(n)} \displaystyle\int_{\mathbb{R}^n} \dfrac{f(y)}{|x-y|^{n-2}\, \mathrm{d}y} & (n \geq 3) \end{cases}
\end{aligned} \tag{10}
$$

will solve Laplace's equation (1) *however, this is wrong*. The estimate (9) indicates that $D^2\Phi(x-y)$ is *not* summable near the singularity at $y = x$, and so naive differentiation through the integral sign is unjustified (and incorrect).

Proceed by carefully calculating Δu. Assume now for simplicity $f \in C_C^2(\mathbb{R}^n)$; i.e., f is twice continuously differentiable, with compact support.

THEOREM 1 (Solving Poisson's equation). *Define u by* (10). *Then*

(i) $u \in C^2(\mathbb{R}^n)$, *and*

(ii) $-\Delta u = f$ *in* $\mathbb{R}^n$.

Consequently (10) provides a formula for a solution of Poisson's equation (2) in $\mathbb{R}^n$.

Interpretation of fundamental solution. It is sometimes written

$$-\Delta\Phi = \delta_0 \quad \text{in } \mathbb{R}^n,$$

δ_0 denoting the Dirac measure on $\mathbb{R}^n$ giving unit mass to the point 0. Adopting this notation, it may be formally computed

$$
\begin{aligned}
-\Delta u(x) &= \int_{\mathbb{R}^n} -\Delta_x \Phi(x-y) f(y)\, \mathrm{d}y \\
&= \int_{\mathbb{R}^n} \delta_x f(y)\, \mathrm{d}y = f(x) \quad (x \in \mathbb{R}^n),
\end{aligned}
$$

in accordance with Theorem 1.

2.2.2 Mean-value formulas.

Consider an open set $U \subset \mathbb{R}^n$ and suppose u is a harmonic function within U. The *mean-value formulas* declare that $u(x)$ equals both the average of u over the sphere $\partial B(x,r)$ and the average of u over the entire ball $B(x,r)$, provided $B(x,r) \subset U$.

THEOREM 2 (Mean-value formulas for Laplace's equation). *If $u \in C^2(U)$ is harmonic, then*

$$u(x) = \fint_{\partial B(x,r)} u \, \mathrm{d}S = \fint_{B(x,r)} u \, \mathrm{d}y \tag{11}$$

for each ball $B(x,r) \subset U$.

THEOREM 3 (Converse to mean-value property). *If $u \in C^2(U)$ satisfies*

$$u(x) = \fint_{\partial B(x,r)} u \, \mathrm{d}S$$

for each ball $B(x,r) \subset U$, then u is harmonic.

2.2.3 Properties of harmonic functions.

The following are a sequence of deductions about harmonic functions that are all based upon the mean-value formulas. Assume for the following that $U \subset \mathbb{R}^n$ is open and bounded.

a. **Strong maximum principle, uniqueness.** A harmonic function must attain its maximum on the boundary and cannot attain its maximum in the interior of a connected region unless it is constant.

THEOREM 4 (Strong maximum principle). *Suppose $u \in C^2(C) \cap C(\bar{U})$ is harmonic within U.*

(i) (Maximum principle for Laplace's equation). *Then*

$$\max_{\bar{U}} = \max_{\partial U} u.$$

(ii) (Strong maximum principle). *Furthermore, if U is connected and there exists a point $x_0 \in U$ such that*

$$u(x_0) = \max_{\bar{U}} u,$$

then

u is constant within U.

Replacing u by $-u$ recovers similar assertions with "min" replacing "max".

Positivity. The strong maximum principle asserts in particular that if U is connected and $u \in C^2(U) \cap C(\bar{U})$ satisfies

$$\begin{cases} \Delta u = 0 & \text{in } U \\ u = g & \text{on } \partial U, \end{cases}$$

where $g \geq 0$, then *u is positive everywhere in U if g is positive somewhere on ∂U*.

An important application of the maximum principle is establishing the uniqueness of solutions to certain boundary-value problems for Poisson's equation.

THEOREM 5 (Uniqueness). *Let $g \in C(\partial U)$, $f \in C(U)$. Then there exists at most one solution $u \in C^2(U) \cap C(\bar{U})$ of the boundary-value problem*

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U. \end{cases} \tag{12}$$

b. Regularity. If $u \in C^2$ is harmonic, then necessarily $u \in C^\infty$. Thus *harmonic functions are automatically infinitely differentiable*. This sort of assertion is called a *regularity* theorem. The interesting point is that the algebraic structure of Laplace's equation $\Delta u = \sum_{i=1}^n u_{x_i x_i} = 0$ leads to the analytic deduction that all the partial derivatives of u exist, even those which do not appear in the PDE.

THEOREM 6 (Smoothness). *If $u \in C(U)$ satisfies the mean-value property (11) for each ball $B(x, r) \subset U$, then*

$$u \in C^\infty(U).$$

Note carefully that u may be smooth, or even continuous, up to ∂U.

c. Local estimates for harmonic functions. The mean-value formulas are now employed to derive careful estimates on the various partial derivaties of a harmonic function. The precise structure of these estimates are needed to prove analyticity.

THEOREM 7 (Estimates on derivatives). *Assume u is harmonic in U. Then*

$$|D^\alpha u(x_0)| \leq \frac{C_k}{r^{n+k}} ||u||_{L^1(B(x,r))} \tag{13}$$

for each ball $B(x_0, r) \subset U$ and each multiindex α of order $|\alpha| = k$. Here

$$C_0 = \frac{1}{\alpha(n)}, \; C_k = \frac{(2^{n+1} n k)^k}{\alpha(n)} \quad (k = 1, \ldots). \tag{14}$$

d. Liouville's Theorem. It is now asserted that there are no nontrivial bounded harmonic functions on all of $\mathbb{R}^n$.

THEOREM 8 (Liouville's Theorem). *Suppose* $u : \mathbb{R}^n \to \mathbb{R}$ *is harmonic and bounded. Then* u *is constant.*

THEOREM 9 (Representation formula). *Let* $f \in C^2_C(\mathbb{R}^n)$, $n \geq 3$. *Then any bounded solution of*

$$-\Delta u = f \quad \textit{in } \mathbb{R}^n$$

has the form

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, \mathrm{d}y + C \quad (x \in \mathbb{R}^n)$$

for some constant C.

Remark. If $n = 2$, $\Phi(x) = -\dfrac{1}{2\pi} \log |x|$ is unbounded as $|x| \to \infty$ and so may be $\int_{\mathbb{R}^n} \Phi(x - y) f(y)\, \mathrm{d}y$.

e. Analyticity. Theorem 6 is now refined:

THEOREM 10 (Analyticity). *Assume is harmonic in* U. *Then* u *is analytic in* U.

f. Harnack's inequality. Writing $V \subset\subset U$ denotes that $V \subset \bar{V} \subset U$ and $\bar{V}$ is compact.

THEOREM 11 (Harnack's inequality). *For each connected open set* $V \subset\subset U$, *there exists a positive constant* C, *depending only on* V, *such that*

$$\sup_V u \leq C \inf_V u$$

for all nonnegative harmonic functions in U.

Thus in particular

$$\frac{1}{C} u(y) \leq u(x) \leq C u(y)$$

for all points $x, y \in V$. These inequalities assert that *the values of a nonnegative harmonic function within* V *are all compatible:* u cannot be very small (or very large) at any point of V unless u is very small (or very large) everywhere in V. The intuitive idea is that since V is apositive distance away from ∂U, there is "room for averaging effects of Laplace's equation to occur".

2.2.4 Green's function.

Assume now $U \subset \mathbb{R}^n$ is open, bounded, and ∂U is C^1. The following proposal is to obtain a general representation formula for the solution of Poisson's equation

$$-\Delta u = f \quad \text{in } U,$$

subject to the prescribed boundary condition

$$u = g \quad \text{on } \partial U.$$

Recall Green's formulas.

THEOREM (Green's formulas). *Let $u, v \in C^2(\bar{U})$. Then*

(i) $\int_U \Delta u \, \mathrm{d}x = \int_{\partial U} \frac{\partial U}{\partial \nu} \, \mathrm{d}S$,

(ii) $\int_U Dv \cdot Du \, \mathrm{d}x = -\int_U u \Delta v \, \mathrm{d}x + \int_{\partial U} \frac{\partial v}{\partial \nu} u \, \mathrm{d}S$,

(iii) $\int_U u\Delta v - v\Delta u \, \mathrm{d}x = \int_{\partial U} u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \, \mathrm{d}S$.

a. **Derivation of Green's function**. Suppose $u \in C^2(\bar{U})$ is an arbitrary function. Fix $x \in U$, choose $\epsilon > 0$ so small that $B(x,\epsilon) \subset U$, and apply the foregoing Green's formula on the region $V_\epsilon := U - B(x,\epsilon)$ to $u(y)$ and $\Phi(y-x)$. Thereby, compute

$$\begin{aligned} &\int_{V_\epsilon} u(y)\Delta\Phi(y-x) - \Phi(y-x)\Delta u(y) \, \mathrm{d}y \\ &= \int_{\partial V_\epsilon} u(y)\frac{\partial \Phi}{\partial \nu}(y-x) - \Phi(y-x)\frac{\partial u}{\partial \nu}(y) \, \mathrm{d}S(y), \end{aligned} \tag{15}$$

ν denoting the outer unit normal vector on ∂V_ϵ. Recall next $\Delta\Phi(x-y) = 0$ for $x \neq y$. Observe also

$$\left| \int_{\partial B(x,\epsilon)} \Phi(y-x)\frac{\partial u}{\partial \nu}(y) \, \mathrm{d}S(y) \right| \leq C\epsilon^{n-1} \max_{\partial B(0,\epsilon)} |\Phi| = o(1)$$

as $\epsilon \to 0$. Furthermore the calculations in the proof of Theorem 1 show

$$\int_{\partial B(x,\epsilon)} u(y)\frac{\partial \Phi}{\partial \nu}(y-x) \, \mathrm{d}S(y) = ⨍_{\partial B(x,\epsilon)} u(y) \, \mathrm{d}S(y) \to u(x)$$

as $\epsilon \to 0$. Hence, sending $\epsilon \to 0$ in 15 yields the formula

$$\begin{aligned} u(x) = &\int_{\partial U} \Phi(y-x)\frac{\partial u}{\partial \nu}(y) - u(y)\frac{\partial \Phi}{\partial \nu}(y-x) \, \mathrm{d}S(y) \\ &- \int_U \Phi(y-x)\Delta u(y) \, \mathrm{d}y. \end{aligned} \tag{16}$$

This identity is valid for any point $x \in U$ and any function $u \in C^2(\bar{U})$.

Now formula (16) would permit solving for $u(x)$ *if* one knew the values of Δu within U and the values of u, $\partial u/\partial \nu$ along ∂U. However, for the application here to Poisson's equation with prescribed boundary values for u, the normal derivative $\partial u/\partial \nu$ along ∂U is unknown. Therefore, (16) must be somehow modified to remove this term.

The idea is now to introduce for fixed x a *corrector* function $\phi^x = \phi^x(y)$, solving the boundary-value problem

$$\begin{cases} \Delta \phi^x = 0 & \text{in } U \\ \phi^x = \Phi(y-x) & \text{on } \partial U. \end{cases} \tag{17}$$

Apply Green's formula once more to compute

$$\begin{aligned} -\int_U \phi^x(y)\Delta u(y)\,\mathrm{d}y &= \int_{\partial U} u(y)\frac{\partial \phi^x}{\partial \nu}(y) - \phi^x(y)\frac{\partial u}{\partial \nu}(y)\,\mathrm{d}S(y) \\ &= \int_{\partial U} u(y)\frac{\partial \phi^x}{\partial \nu}(y) - \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,\mathrm{d}S(y). \end{aligned} \tag{18}$$

DEFINITION. *Green's function* for the region U is

$$G(x,y) := \Phi(y-x) - \phi^x(y) \quad (x, y \in U,\ x \neq y).$$

Adopting this terminology and adding (18) to (16) yields

$$u(x) = -\int_{\partial U} u(y)\frac{\partial G}{\partial \nu}(x,y)\,\mathrm{d}S(y) - \int_U G(x,y)\Delta u(y)\,\mathrm{d}y \quad (x \in U), \tag{19}$$

where

$$\frac{\partial G}{\partial \nu}(x,y) = D_y G(x,y)\cdot \nu(y)$$

is the outer normal derivative of G with respect to the variable y. Observe that the term $\partial u/\partial \nu$ does not appear in the equation (19): the corrector ϕ^x was introduced precisely to achieve this.

Suppose now $u \in C^2(\bar{U})$ solves the boundary-value problem

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U, \end{cases} \tag{20}$$

for given continuous functions f, g. Plugging into (18), one obtains

THEOREM 12 (Representation formula using Green's function). *If $u \in C^2(\bar{U})$ solves the problem* (20)*, then*

$$u(x) = -\int_{\partial U} g(y)\frac{\partial G}{\partial \nu}(x,y)\,\mathrm{d}S(y) + \int_U f(y)G(x,y)\,\mathrm{d}y \quad (x \in U). \tag{21}$$

(21) is a formula for the solution of the boundary-value problem (20), provided one can construct Green's function G for the given domain U. This is in general a difficult matter and can be done only when U has simple geometry. Subsequent subsections below identify some special cases for which an explicit calculation of G is possible.

Interpreting Green's function. Fix $x \in U$. Then regarding G as a function of y, symbolically one may write

$$\begin{cases} -\Delta G = \delta_x & \text{in } U \\ \quad G = 0 & \text{on } \partial U, \end{cases}$$

δ_x denoting the Dirac measure giving unit mass to the point x.

Note the general assertion that G is symmetric in the variables x and y:

THEOREM 13 (Symmetry of Green's function). *For all $x,\ y \in U,\ x \neq y$, one has*

$$G(y, x) = G(x, y).$$

In the next two subsections, Green's functions will be built for two regions with simple geometry, namely the half space $\mathbb{R}^n_+$ and the unit bell $B(0, 1)$. Everything depends upon explicitly solving the corrector problem (17) in these regions, and this in turn depends upon some clever geometric reflection tricks.

b. Green's function for a half space. First consider the half-space

$$\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\}.$$

Although this region is unbounded, and so the calculations in the previous section do not apply, it will be attempted nevertheless to build Green's function using the ideas developed earlier. Later of course one must check directly that the corresponding representation formula is valid.

DEFINITION. If $x = (x_1, \dots, x_{n-1}, -x_n) \in \mathbb{R}^n_+$, its *reflection* in the plane $\partial\mathbb{R}^n_+$ is the point

$$\tilde{x} = (x_1, \dots, x_{n-1}, -x_n).$$

The problem (17) will be solved for the half-space by setting

$$\phi^x(y) := \Phi(y - \tilde{x}) = \Phi(y_1 - x_1, \dots, y_{n-1} - x_{n-1}, y_n + x_n) \quad (x, y \in \mathbb{R}^n_+).$$

The idea is that the corrector ϕ^x is built from Φ by "reflecting the singularity" from $x \in \mathbb{R}^n_+$ to $\tilde{x} \notin \mathbb{R}^n_+$.

Note that

$$\phi^x(y) = \Phi(y - x) \quad \text{if } y \in \partial\mathbb{R}^n_+,$$

and thus

$$\begin{cases} \Delta\phi^x = 0 & \text{in } \mathbb{R}^n_+ \\ \phi^x = \Phi(y - x) & \text{on } \partial\mathbb{R}^n_+, \end{cases}$$

as required.

DEFINITION. *Green's function for the half-space $\mathbb{R}^n_+$ is*

$$G(x, y) := \Phi(y - x) - \Phi(y - \tilde{x}) \quad (x, yin\mathbb{R}^n_+,\ x \neq y).$$

Then

$$\begin{aligned} G_{y_n}(x, y) &= \Phi_{y_n}(y - x) - \Phi_{y_n}(y - \tilde{x}) \\ &= \frac{-1}{n\alpha(n)}\left[\frac{y_n - x_n}{|y - x|^n} - \frac{y_n + x_n}{|y - \tilde{x}|^n}\right]. \end{aligned}$$

Consequently if $y \in \partial\mathbb{R}^n_+$,

$$\frac{\partial G}{\partial \nu}(x, y) = -G_{y_n}(x, y) = -\frac{2x_n}{n\alpha(n)}\frac{1}{|x - y|^n}.$$

Suppose now u solves the boundary-value problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \\ u = q & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{22}$$

Then from (21) we expect

$$u(x) = \frac{2x_n}{n\alpha(n)}\int_{\partial\mathbb{R}^n_+}\frac{g(y)}{|x - y|^n}\,\mathrm{d}y \quad (x \in \mathbb{R}^n_+) \tag{23}$$

to be a representation formula for our solution. The function

$$K(x, y) := \frac{2x_n}{n\alpha(n)}\frac{1}{|x - y|^n} \quad (x \in \mathbb{R}^n_+,\ y \in \partial\mathbb{R}^n_+)$$

is *Poisson's kernel* for $\mathbb{R}^n_+$, and (23) is *Poisson's formula*.

One must now check directly that formula (23) does indeed provide a solution of the boundary-value problem (22).

THEOREM 14 (Poisson's formula for half-space). *Assume* $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$, *and define by* (23). *Then*

(i) $u \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$

(ii) $\Delta u = 0 \quad$ *in* $\mathbb{R}^n_+$, *and*

(iii) $\lim_{\substack{x \to x^0 \\ x \in \mathbb{R}^n_+}} u(x) = g(x^0) \quad$ *for each point* $x^0 \in \partial\mathbb{R}^n_+$.

c. **Green's function for a ball.** To construct Green's function for the unit ball $B(0,1)$, once again employ a kind of reflection, this time through the sphere $\partial B(0,1)$.

DEFINITION. If $x \in \mathbb{R}^n - \{0\}$, the point

$$\tilde{x} = \frac{x}{|x|^2}$$

is called the point *dual* to x with respect to $\partial B(0,1)$. The mapping $x \mapsto \tilde{x}$ is *inversion* through the unit sphere $\partial B(0,1)$.

The inversion is now employed through the sphere to compute Green's function for the unit ball $U = B^0(0,1)$. Fix $x \in B^0(0,1)$. Remember that one must find a corrector function $\phi^x = \phi^x(y)$ solving

$$\begin{cases} \Delta \phi^x = 0 & \text{in } B^0(0,1) \\ \phi^x = \Phi(y-x) & \text{on } \partial B(0,1); \end{cases} \tag{24}$$

then Green's function will be

$$G(x,y) = \Phi(y-x) - \phi^x(y). \tag{25}$$

The idea now is to "invert the singularity" from $x \in B^0(0,1)$ to $\tilde{x} \notin B(0,1)$. Assume for the moment $n \geq 3$. Now the mapping $y \mapsto \Phi(y-\tilde{x})$ is harmonic for $y \neq \tilde{x}$. Thus $y \mapsto |x|^{2-n}\Phi(y-\tilde{x}$ is harmonic for $y \neq \tilde{x}$, and so

$$\phi^x(y) := \Phi(|x|(y-\tilde{x})) \tag{26}$$

is harmonic in U. Furthermore, if $y \in \partial B(0,1)$ and $x \neq 0$,

$$\begin{aligned} |x|^2|y-\tilde{x}|^2 &= |x|^2\left(|y|^2 - \frac{2y\cdot x}{|x|^2} + \frac{1}{|x|^2}\right) \\ &= |x|^2 - 2y\cdot x + 1 = |x-y|^2. \end{aligned}$$

Thus $(|x||y-\tilde{x}|^{-(n-2)} = |x-y|^{-(n-2)}$. Consequently

$$\phi^x(y) = \Phi(y-x) \quad (y \in \partial B(0,1)), \tag{27}$$

as required.

DEFINITION. *Green's function for the unit ball is*

(28) $$G(x,y) := \Phi(y-x) - \Phi(|x|(y-\tilde{x})) \quad (x,y \in B(0,1),\ x \neq y).$$

The same formula is valid for $n = 2$ as well.

Assume now u solves the boundary-value problem

(29) $$\begin{cases} \Delta u = 0 & \text{in } B^0(0,1) \\ u = g & \text{on } \partial B(0,1). \end{cases}$$

Then using (21), we see

(30) $$u(x) = -\int_{\partial B(0,1)} g(y) \frac{\partial G}{\partial \nu}(x,y)\, \mathrm{d}S(y).$$

According to (28),

$$G_{y_i}(x,y) = \Phi_{y_i}(y-x) - \Phi(|x|(y-\tilde{x}))_{y_i}.$$

But

$$\Phi_{y_i}(y-x) = \frac{1}{n\alpha(n)} \frac{x_i - y_i}{|x-y|^n},$$

and furthermore

$$\Phi(|x|(y-\tilde{x}))_{y_i} = \frac{-1}{n\alpha(n)} \frac{y_i|x|^2 - x_i}{(|x||y-\tilde{x}|)^n} = -\frac{1}{n\alpha(n)} \frac{y_i|x|^2 - x_i}{|x-y|^n}$$

if $y \in \partial B(0,1)$. Accordingly

$$\begin{aligned} \frac{\partial G}{\partial \nu}(x,y) &= \sum_{i=1}^n y_i G_{y_i}(x,y) \\ &= \frac{-1}{n\alpha(n)} \frac{1}{|x-y|^n} \sum_{i=1}^n y_i((y_i - x_i) - y_i|x|^2 + x_i) \\ &= \frac{-1}{n\alpha(n)} \frac{1 - |x|^2}{|x-y|^n}. \end{aligned}$$

Hence formula (30) yields the representation formula

$$u(x) = \frac{1-|x|^2}{n\alpha(n)} \int_{\partial B(0,1)} \frac{g(y)}{|x-y|^n}\, \mathrm{d}S(y).$$

Suppose now instead of (29) u solves the boundary-value problem

(31) $$\begin{cases} \Delta u = 0 & \text{in } B^0(0,r) \\ u = g & \text{on } \partial B(0,r) \end{cases}$$

for $r > 0$. Then $\tilde{u}(x) = u(rx)$ solves (29), with $\tilde{g}(x) = g(rx)$ replacing g. We change variables to obtain *Poisson's formula*

(32) $$u(x) = \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\, \mathrm{d}S(y) \quad (x \in B^0(0,r)).$$

The function

$$K(x,y) := \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{1}{|x-y|^n} \quad (x \in B^0(0,r),\ y \in \partial B(0,r))$$

is *Poisson's kernel* for the ball $B(0,r)$.

Thus (32) has been established under the assumption that a smooth solution of (31) exists. Next, it is asserted that this formula in fact gives a solution:

THEOREM 15 (Poisson's formula for a ball). *Assume* $g \in C(\partial B(0,r))$ *and define* u *by* (32). *Then*

(i) $u \in C^\infty(B^0(o,r))$,

(ii) $\Delta u = 0$ *in* $B^0(0,r)$, *and*

(iii) $\lim_{\substack{x \to x^0 \\ x \in B^0(0,r)}} u(x) = g(x^0)$ *for each point* $x^0 \in \partial B(0,r)$.

Note that the proof of Theorem 15 is similar to that for Theorem 14.

2.2.5 Energy methods.

Most of the analysis of harmonic functions thus far has depended upon fairly explicit representation formulas entailing the fundamental solution, Green's functions, etc. This concluding section illustrates some "energy methods", i.e., techniques involving the L^2-norms of various expressions.

a. Uniqueness. Consider first the boundary-value problem

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U. \end{cases} \tag{33}$$

The maximum principle was already employed in §2.2.3 to show uniqueness, but now a simple alternative proof is set forth. Assume U is open, bounded, and ∂U is C^1.

THEOREM 16 (Uniqueness). *There exists at most one solution* $u \in C^2(\bar{U})$ *of* (33).

b. Dirichlet's principle. Next, a solution is demonstrated of the boundary-value problem (33) for Poisson's equation that can be characterized as the minimizer of an appropriate functional. For this, define the *energy functional*

$$I[w] := \int_U \frac{1}{2}|Dw|^2 - wf \, \mathrm{d}x,$$

w belonging to the *admissible set*

$$\mathcal{A} := \left\{ w \in C^2(\bar{U}) \mid w = g \text{ on } \partial U \right\}.$$

THEOREM 17 (Dirichlet's principle). *Assume $u \in C^2(\bar{U})$ solves* (33). *Then*

$$I[u] = \min_{w \in \mathcal{A}} [w]. \tag{34}$$

Conversely, if $u \in \mathcal{A}$ satisfies (34)*, then u solves the boundary-value problem* (33).

In other words if $u \in \mathcal{A}$, the PDE $-\Delta u = f$ is equivalent to the statement that u minimizes the energy $I[\cdot]$ among functions in $\mathcal{A}$.

Dirichlet's principle is an instance of the *calculus of variations* applied to Laplace's equation.