Classical Electromagnetism Notes
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1 Calculus

1.1 Vector derivatives

Change of variables

$$\text{Rectangular/Cartesian: } d\underline{l} = dx\ \underline{\hat{x}} + dy\ \underline{\hat{y}} + dz\ \underline{\hat{z}} \tag{1}$$

$$\text{Spherical: } d\underline{l} = dr\ \underline{\hat{r}} + r\ d\theta\ \underline{\hat{\theta}} + r\sin\theta\ d\phi\ \underline{\hat{\phi}} \tag{2}$$

$$\text{Cylindrical: } d\underline{l} = ds\ \underline{\hat{s}} + s\ d\phi\ \underline{\hat{\phi}} + dz\ \underline{\hat{z}} \tag{3}$$

Gradient

$$\text{Rectangular/Cartesian: } \nabla t = \frac{\partial t}{\partial x}\ \underline{\hat{x}} + \frac{\partial t}{\partial y}\ \underline{\hat{y}} + \frac{\partial t}{\partial z}\ \underline{\hat{z}} \tag{4}$$

$$\text{Spherical: } \nabla t = \frac{\partial t}{\partial r}\ \underline{\hat{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\ \underline{\hat{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\ \underline{\hat{\phi}} \tag{5}$$

$$\text{Cylindrical: } \nabla t = \frac{\partial t}{\partial s}\ \underline{\hat{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\ \underline{\hat{\phi}} + \frac{\partial t}{\partial z}\ \underline{\hat{z}} \tag{6}$$

Divergence

$$\text{Rectangular/Cartesian: } \nabla \cdot \underline{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \tag{7}$$

$$\text{Spherical: } \nabla \cdot \underline{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} \tag{8}$$

$$\text{Cylindrical: } \nabla \cdot \underline{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z} \tag{9}$$

Curl

$$\text{Rectangular/Cartesian: } \nabla \times \underline{v} = (\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z})\ \underline{\hat{x}} + (\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x})\ \underline{\hat{y}} + (\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y})\ \underline{\hat{z}} \tag{10}$$

$$\text{Spherical: } \nabla \times \underline{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\ v_\phi\right) - \frac{\partial v_\theta}{\partial\phi}\right]\underline{\hat{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}\left(r v_\phi\right)\right] + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r v_\theta\right) - \frac{\partial v_r}{\partial\theta}\right]\underline{\hat{\phi}} \tag{11}$$

$$\text{Cylindrical: } \nabla \times \underline{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\underline{\hat{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\underline{\hat{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}\left(s v_\phi\right) - \frac{\partial v_s}{\partial\phi}\right]\underline{\hat{z}} \tag{12}$$

Laplacian

$$\text{Rectangular/Cartesian: } \nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} \tag{13}$$

$$\text{Spherical: } \nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial t}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial t}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2} \tag{14}$$

$$\text{Cylindrical: } \nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}(s\frac{\partial t}{\partial s}) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2} \tag{15}$$

$$\tag{16}$$

1.2 Evans PDE derivatives notes

Given a vector (multiindex) $\alpha = (\alpha_1, \ldots, \alpha_n)$ of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$, define

$$D^\alpha u(x) = \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}. \tag{17}$$

If k is a nonnegative integer,

$$D^k u(x) := \{D^\alpha u(x) \mid |\alpha| = k\}, \tag{18}$$

the set of all partial derivatives of order k. Assigning some ordering to the various partial derivatives, we can also regard $D^k u(x)$ as a point in $\mathbb{R}^{n^k}$.

$$|D^k u| = \left(\sum_{|\alpha|=k} |D^\alpha u|^2 \right)^{1/2} \tag{19}$$

If $k = 1$, we regard the elements of Du as being arranged in a vector

$$Du := (u_{x_1}, \ldots, u_{x_n}) = \text{gradient vector } \Rightarrow \ Du \in \mathbb{R}^n. \tag{20}$$

Sometimes $u_r := \frac{x}{|x|} \cdot Du$ $(x \neq 0)$ denotes the radial derivative of u.
If $k = 2$, the elements of $D^2 u$ are regarded as being arranged in a matrix

$$D^2 u := \begin{pmatrix} u_{x_1x_1} & \cdots & u_{x_1x_n} \\ \vdots & \ddots & \vdots \\ u_{x_nx_1} & \cdots & u_{x_nx_n} \end{pmatrix} = \text{Hessian matrix } \Rightarrow D^2 u \in \mathbb{S}^n, \text{ the space of real symmetric } n \times n \text{ matrices.} \tag{21}$$

In this notation, the Laplacian of u is denoted $\Delta u = \sum_{i=1}^n u_{x_ix_i}$.
Thus, $\Delta u = \text{div } Du = \text{tr}(D^2 u)$. Sometimes an attached subscript to the symbols D, D^2, etc. is employed to denote the variables being differentiated, e.g., if $u = u(x, y)$ such that $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, then $D_x u = (u_{x_1}, \ldots, u_{x_n})$, $D_y u = (u_{y_1}, \ldots, u_{y_n})$.

1.2.1 Vector-valued Functions

If now $m > 1$ and $\underline{u}: U \to \mathbb{R}^m$, $\underline{u} = (u^1, \ldots, u^m)$, define

$$D^\alpha \underline{u} = (D^\alpha u^1, \ldots, D^\alpha u^m) \tag{22}$$

for each multiindex α. Then

$$D^k \underline{u} = \{D^\alpha \underline{u} \mid |\alpha| = k\} \tag{23}$$

and

$$|D^k \underline{u}| := \left(\sum_{|\alpha|=k} |D^\alpha \underline{u}|^2 \right)^{1/2}. \tag{24}$$

In the special case $k = 1$, we write

$$D\underline{u} := \begin{pmatrix} u^1_{x_1} & \cdots & u^1_{x_n} \\ \vdots & \ddots & \vdots \\ u^m_{x_1} & \cdots & u^m_{x_n} \end{pmatrix} = \text{ gradient matrix} \tag{25}$$

If $m = n$, we have

$$\div \underline{u} := \operatorname{tr}(D\underline{u}) = \sum_{i=1}^n u^i_{x_i} = \text{ divergence of } \underline{u} \tag{26}$$

The spaces $C(U;\mathbb{R}^m)$, $\mathcal{L}^p(U;\mathbb{R}^m)$, etc. consist of those functions $\underline{u}: U \to \mathbb{R}^m$, $\underline{u} = (u^1, \ldots, u^m)$, with $u^i \in C(U), \mathcal{L}^p(U)$, etc. where $i = 1, \ldots, m$.

1.2.2 Boundaries

Let $U \in \mathbb{R}^n$ be open and bounded, $k = \{1, 2, \ldots\}$.

Definition 1.1. The **boundary** ∂U is C^k if for each point $x^0 \in \partial U \exists r > 0 \& C^k$ function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that–upon relabeling and reorienting the coordinate axes if necessary–we have

$$U \cap B(x^0, r) = \{x \in B(x^0, r) \big| x_n > \gamma(x_1, \ldots, x_{n-1}\}. \tag{27}$$

Likewise, ∂U is C^∞ if ∂U is C^k for $k = 1, 2, \ldots$, and ∂U is **analytic** if the mapping γ is analytic.

Definition 1.2. If ∂U is C^1, then along ∂U is defined the outward pointing **unit normal vector field**

$$\upsilon = (\upsilon^1, \ldots, \upsilon^n). \tag{28}$$

The **unit normal** at any point $x^0 \in \partial U$ is $\upsilon(x^0) = \upsilon = (\upsilon_1, \ldots, \upsilon_n)$.

Definition 1.3. Let $u \in C^1(\bar{U})$. We call

$$\frac{\partial u}{\partial \upsilon} := \underline{\upsilon} \cdot Du \tag{29}$$

the **(outward) normal derivative** of u.

1.2.3 Function Spaces Notation

$$C(U) = \{u :\ U \to \mathbb{R} \mid u \text{ continuous}\} \tag{30}$$
$$C(\bar{U}) = \{u \in C(U) \mid u \text{ is uniformly continuous on bounded subsets of } U\} \tag{31}$$
$$C^k(U) = \{u :\ U \to \mathbb{R} \mid u \text{ is k-times continuously differentiable}\} \tag{32}$$
$$C^k(\bar{U}) = \{u \in C^k(U) \mid D^\alpha \text{ is uniformly continuous on bounded subsets of } U,\ \forall\ |\alpha| \leq k\} \tag{33}$$
$$C^\infty(U) = \{u :\ U \to \mathbb{R} \mid u \text{ is infinitely differentiable}\} = \bigcap_{k=0}^{\infty} C^k(U) \tag{34}$$
$$C^\infty(\bar{U}) = \bigcup_{k=0}^{\infty} C^k(\bar{U}) \tag{35}$$
$$\mathcal{L}^p(U) = \{u :\ U \to \mathbb{R} \mid u \text{ is Lebesgue measurable, } \|u\|_{\mathcal{L}^p(U)} < \infty\}, \tag{36}$$
$$\text{where } \|u\|_{\mathcal{L}^p(u)} < \infty \ := \ \left(\int_U |u|^p \mathrm{d}x\right)^{1/p} \tag{37}$$
$$\mathcal{L}^\infty(u) = \{u :\ U \to \mathbb{R} \mid u \text{ is Lebesgue measurable, } \|u\|_{\mathcal{L}^\infty(U)} < \infty\}, \tag{38}$$
$$\text{where } \|u\|_{\mathcal{L}^\infty(u)} < \infty \ := \ \operatorname*{ess\,sup}_U |u| \tag{39}$$

$\operatorname{ess\,sup} \equiv$ essential supremum, of a function is the smallest value that is greater than or equal to the function values everywhere while ignoring what the function does at a set of points of measure zero.

1.2.4 Theorems and Formulas

Theorem 1.1 (Gauss-Green Theorem)**.** Assume U is a bounded, open subset of $\mathbb{R}^n$ and ∂U is C^1.

i) Suppose $u \in C^1(\bar{U})$. Then $\int_U u_{x_i} \mathrm{d}x = \int_{\partial U} uv^i \ \mathrm{d}S \quad (i = 1, \ldots, n)$.

ii) **Divergence Theorem.** We have $\int_U \operatorname{div}\ \underline{u}\ \mathrm{d}x = \int_{\partial U} \underline{u} \cdot v\ \mathrm{d}S$ for each vector field $\underline{u} \in C^1(\bar{U}; \mathbb{R}^n)$. (ii) follows from (i) applied to each component of $\underline{u} = (u^1, \ldots, u^n)$.

Theorem 1.2 (Integration by parts formula)**.** Let $U, V \in C^1(\bar{U})$. Then $\int_U u_{x_i} \mathrm{d}x = -\int_U uv_{x_i}\ \mathrm{d}x + \int_{\partial U} uvv^i\ \mathrm{d}S \quad (i = 1, \ldots, n)$
Proof: Apply (i) to uv.

Theorem 1.3 (Green's Formulas)**.** Let $u, v \in C^2(\bar{U})$. Then

i) $\int_U \Delta u\ \mathrm{d}x = \int_{\partial U} \frac{\partial u}{\partial v}\ \mathrm{d}S$,

ii) $\int_U Dv \cdot Du\ \mathrm{d}x = -\int_U u\Delta v\ \mathrm{d}x + \int_{\partial U} \frac{\partial u}{\partial v} u \mathrm{d}S$,

iii) $\int_U u\Delta v - v\Delta u\ \mathrm{d}x = \int_{\partial U} u\frac{\partial u}{\partial v} - v\frac{\partial u}{\partial v}\ \mathrm{d}S$.

In Evan's PDE, denote

$$\text{Laplace's Equation: } \Delta u = 0 \tag{40}$$
$$\text{Poisson's Equation: } -\Delta u = f \tag{41}$$

where the negative sign on the Laplacian in Poisson's Equation is to be consistent with the notation he uses for general 2^{nd}-order elliptic operators.

1.2.5 Boundary conditions

Dirichlet boundary conditions ("first kind")

- Specify the value of the solution u itself on the boundary $\partial\Omega$: $u(\underline{x}) = g(\underline{x})$ for all $\underline{x} \in \partial\Omega$, where g is given.
- Standard for boundary value problems.
- Typically for elliptic and parabolic PDEs.

von Neumann boundary conditions ("second kind")

- Specify the normal derivative (gradient in the outward normal direction) of the solution on the boundary: $\frac{\partial u}{\partial n}(\underline{x}) = \underline{n} \cdot \nabla u(\underline{x}) = h(\underline{x})$ for all $\underline{x} \in \partial\Omega$, where h is given.
- Physically: flux, heat flow, or stress across the boundary.
- Typically for elliptic and parabolic PDEs.

Cauchy boundary conditions

- Specify both the value of the function and its normal derivative on the boundary: $u(\underline{x}) = g(\underline{x})$ and $\frac{\partial u}{\partial n}(\underline{x}) = h(\underline{x})$ for all $\underline{x} \in \partial\Omega$.
- Relevant for hyperbolic PDEs (e.g. wave equation) on open (non-characteristic) hypersurfaces $\Rightarrow$ well-posed IVP.
- For elliptic PDEs, Cauchy conditions on a closed boundary are typically overdetermined and lead to ill-posed problems.
- In time-dependent problems, Cauchy conditions often appear as initial conditions (e.g. at $t = 0$), distinct from spatial boundary conditions.
- Distinct from Robin boundary conditions ("third kind"): $\alpha u + \beta \frac{\partial u}{\partial n} = k$.

2 Electrostatics

$$\underline{F} = q\underline{E} \qquad (42)$$

Electric field at a point charge $\Rightarrow$ Force

$$\underline{F} = kq_1q_2\frac{\underline{x}_1 - \underline{x}_2}{|\underline{x}_1 - \underline{x}_2|^3} \qquad (43)$$

Coulomb's Law; F at a point charge q_1 at $\underline{x}_1$

$$\underline{E}(\underline{x}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n} q_i\frac{\underline{x} - \underline{x}_i}{|\underline{x} - \underline{x}_i|^3} \qquad (44)$$

The electric field at $\underline{x}$ due to a system of point charges q_i located at $\underline{x}_i$, $i = 1, \ldots, n$. Due to experimentally observed linear superposition of forces due to many charges.

$$\underline{E}(\underline{x}) = \frac{1}{4\pi\epsilon_0}\int \rho(\underline{x}')\frac{\underline{x} - \underline{x}'}{|\underline{x} - \underline{x}'|^3}\,\mathrm{d}^3x' \qquad (45)$$

Charges are so small and so numerous that they can be described by a charge density $\rho(\underline{x}')$, and the previous sum becomes an integral.

If Δq is the charge in a small volume $\Delta x\Delta y\Delta z$ at the point $\underline{x}'$, then $\Delta q = \rho(\underline{x}')\Delta x\Delta y\Delta z$. $\mathrm{d}^3x' = \mathrm{d}x'\mathrm{d}y'\mathrm{d}z'$ is the $3D$ volume element at $\underline{x}'$.

$$\underline{E}\cdot\underline{n}\;\mathrm{d}\theta = \frac{q}{4\pi\epsilon_0}\frac{\cos\theta}{r^2}\mathrm{d}a \qquad (46)$$

Gauss's Law

$$\underline{F} = q\underline{E} \qquad (47)$$

Electric field at a point charge $\Rightarrow$ Force

$$\underline{F} = kq_1q_2\frac{\underline{x}_1 - \underline{x}_2}{|\underline{x}_1 - \underline{x}_2|^3} \qquad (48)$$

Coulomb's law; F at pt. charge q_2 at $\underline{x}_1$

$$\underline{E}(\underline{x}) = kq_1\frac{\underline{x} - \underline{x}_1}{|\underline{x} - \underline{x}_1|^3} \qquad (49)$$

The electric field at $\underline{x}$ due to a pt. charge q_1 at $\underline{x}_1$.

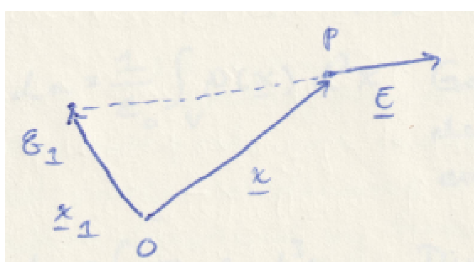

Figure 1: Field at observation point: origin O, position $\underline{x}_1$ of charge q_1, observation point P at $\underline{x}$; $\underline{E}$ is along $\underline{x} - \underline{x}_1$.

$$\underline{E}(\underline{x}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n} q_i\frac{\underline{x} - \underline{x}_i}{|\underline{x} - \underline{x}_i|^3} \qquad (50)$$

The electric field at $\underline{x}$ due to a system of pt. charges q_i at $\underline{x}_i$, $i = 1, \ldots, n$. Due to linear superposition.

$$\underline{E}(\underline{x}) = \frac{1}{4\pi\epsilon_0}\int \rho(\underline{x}')\frac{\underline{x} - \underline{x}'}{|\underline{x} - \underline{x}'|^3}\,\mathrm{d}^3x' \qquad (51)$$

Charges described by charge density $\rho(\underline{x}')$; sum becomes integral.

If Δq is the charge in a small volume $\Delta x\,\Delta y\,\Delta z$ at $\underline{x}'$, then $\Delta q = \rho(\underline{x}')\,\Delta x\,\Delta y\,\Delta z$. The 3D volume element at $\underline{x}'$ is $\mathrm{d}^3x' = \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'$.

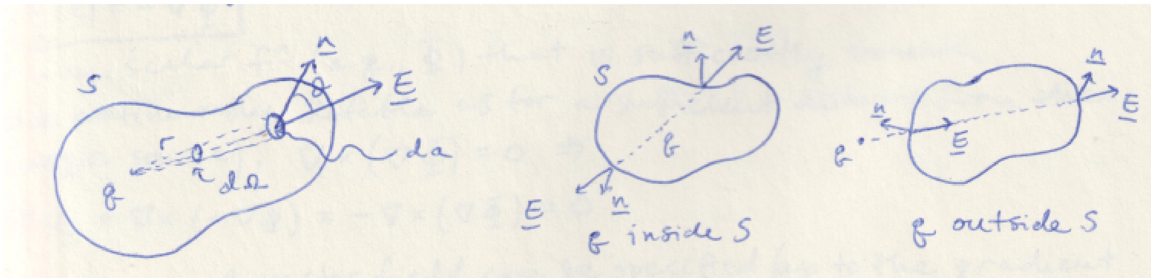

Figure 2: Gauss's Law: flux through area element da (solid angle dΩ); charge q inside S; charge q outside S. Normal $\underline{n}$, field $\underline{E}$.

$$\underline{E} \cdot \underline{n}\, \mathrm{d}a = \frac{q}{4\pi\epsilon_0} \frac{\cos\theta}{r^2}\, \mathrm{d}a \qquad \textbf{Gauss's Law} \tag{52}$$

$\underline{E}$ is directed along the line from the surface element to the charge $q \Rightarrow \cos\theta\, \mathrm{d}a = r^2\, \mathrm{d}\Omega$ where dΩ is the element of solid angle subtended by da at the position of the charge.

$$\oint_S \underline{E} \cdot \underline{n}\, \mathrm{d}a = \begin{cases} q/\epsilon_0, & \text{charge inside } S \\ 0, & \text{charge outside } S \end{cases} \qquad \text{Gauss's Law for a single pt. charge} \tag{53}$$

$$\oint_S \underline{E} \cdot \underline{n}\, \mathrm{d}a = \frac{1}{\epsilon_0} \sum_i q_i \qquad \text{Gauss's Law for discrete charges inside } S \tag{54}$$

$$\oint_S \underline{E} \cdot \underline{n}\, \mathrm{d}a = \frac{1}{\epsilon_0} \int_V \rho(\underline{x})\, \mathrm{d}^3x \qquad \text{Gauss's Law for continuous } \rho(\underline{x});\ V = \text{volume enclosed by } S. \tag{55}$$

Divergence theorem: $\oint_S \underline{A} \cdot \underline{n}\, \mathrm{d}a = \int_V \nabla \cdot \underline{A}\, \mathrm{d}^3x$. Setting the integrand to zero:

$$\boxed{\nabla \cdot \underline{E} = \frac{\rho}{\epsilon_0}}$$

Differential form of Gauss's Law (electrostatics).

Electric field from ρ:

$$\underline{E}(\underline{x}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\underline{x}') \frac{\underline{x} - \underline{x}'}{|\underline{x} - \underline{x}'|^3}\, \mathrm{d}^3x' = -\frac{1}{4\pi\epsilon_0} \int_V \rho(\underline{x}')\, \nabla \left(\frac{1}{|\underline{x} - \underline{x}'|} \right) \mathrm{d}^3x' = -\nabla \left(\frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\underline{x}')}{|\underline{x} - \underline{x}'|}\, \mathrm{d}^3x' \right).$$

Define the scalar potential

$$\Phi(\underline{x}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\underline{x}')}{|\underline{x} - \underline{x}'|}\, \mathrm{d}^3x'.$$

Then $\boxed{\underline{E}(\underline{x}) = -\nabla\Phi}$.

For any sufficiently smooth scalar f (e.g. Φ), $\nabla \times (\nabla f) = 0 \Rightarrow \nabla \times \underline{E} = \nabla \times (-\nabla\Phi) = 0$. Hence

$$\boxed{\nabla \times \underline{E} = 0}.$$

A vector field can be specified (up to the gradient of a scalar satisfying Laplace's eqn.) if its divergence and curl are given everywhere.

Work, path independence, Stokes

Work done on a test charge q moving from A to B: $\underline{F} = q\underline{E}$, $\underline{E} = -\nabla\Phi$, so

$$W = -\int_A^B \underline{F}\cdot \mathrm{d}\underline{l} = -q\int_A^B \underline{E}\cdot \mathrm{d}\underline{l}, \qquad \int_A^B \underline{E}\cdot \mathrm{d}\underline{l} = -(\Phi_B - \Phi_A)$$

(path independent). Net work done on the charge against the field.

If the path is closed, Coulomb's Law $\Rightarrow \oint \underline{E}\cdot \mathrm{d}\underline{l} = 0$.

Work against the field for a general charge distribution (path C from A to B):

$$W_{A\to B}^{\text{against field}} = -q\int_C \underline{E}\cdot \mathrm{d}\underline{l} = -\frac{q}{4\pi\epsilon_0}\int_C \left(\int_V \rho(\underline{x}')\frac{\underline{x}-\underline{x}'}{|\underline{x}-\underline{x}'|^3}\,\mathrm{d}^3x'\right)\cdot \mathrm{d}\underline{l}.$$

By Fubini (e.g. bounded support or sufficient decay), interchange order. For fixed $\underline{x}'$, the inner line integral is the work against the pure Coulomb field of a pt. charge at $\underline{x}'$ from A to B. Since

$$\frac{\underline{x}-\underline{x}'}{|\underline{x}-\underline{x}'|^3} = \nabla\left(\frac{1}{|\underline{x}-\underline{x}'|}\right)$$

, the fundamental theorem for line integrals gives

$$\int_C \frac{\underline{x}-\underline{x}'}{|\underline{x}-\underline{x}'|^3}\cdot \mathrm{d}\underline{l} = \frac{1}{|\underline{a}-\underline{x}'|} - \frac{1}{|\underline{b}-\underline{x}'|}$$

($\underline{a}$, $\underline{b}$ endpoints). Hence

$$W_{A\to B}^{\text{against field}} = -\frac{q}{4\pi\epsilon_0}\int_V \rho(\underline{x}')\left(\frac{1}{|\underline{a}-\underline{x}'|} - \frac{1}{|\underline{b}-\underline{x}'|}\right)\mathrm{d}^3x'.$$

For a closed path ($\underline{a} = \underline{b}$):

$$W_{\text{closed}}^{\text{against field}} = 0 \Rightarrow -q\oint_C \underline{E}\cdot \mathrm{d}\underline{l} = 0 \Rightarrow \oint \underline{E}\cdot \mathrm{d}\underline{l} = 0$$

. By Stokes's theorem,

$$\oint \underline{A}\cdot \mathrm{d}\underline{l} = \int_S (\nabla\times\underline{A})\cdot\underline{n}\,\mathrm{d}a,$$

so $\nabla\times\underline{E} = 0$ in regions where $\underline{E}$ is differentiable (away from charges).

Laplacian of $1/r$, regularized kernels

Consider the family of regularized kernels

$$G_a(r) = \frac{1}{\sqrt{r^2+a^2}} \text{ with } a > 0.$$

This is C^∞ everywhere; as $a \to 0^+$, $G_a(r) \to 1/r$ in appropriate weak senses and pointwise for $r \neq 0$.

The proposed non-regularized solution is

$$\Phi(\underline{x}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\underline{x}')}{|\underline{x}-\underline{x}'|}\,\mathrm{d}^3x'.$$

Applying ∇^2 in $\underline{x}$ and interchanging the integral:

$$\nabla^2\Phi(\underline{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\underline{x}')\,\nabla^2\left(\frac{1}{r}\right)\mathrm{d}^3x', \qquad r = |\underline{x}-\underline{x}'|.$$

Note: $1/r$ is not twice differentiable at $r = 0$; its second derivatives blow up like $\sim 1/r^3$ near $r = 0$ (non-integrable).

Let $r = |\underline{r}| = \sqrt{x^2 + y^2 + z^2}$. For $r \neq 0$ (Cartesian coords.):

$$\partial_x \left(\frac{1}{r}\right) = -\frac{x}{r^3}, \quad \partial_y \left(\frac{1}{r}\right) = -\frac{y}{r^3}, \quad \partial_z \left(\frac{1}{r}\right) = -\frac{z}{r^3}.$$

Second derivatives (e.g. ∂_{xx}):

$$\partial_{xx} \left(\frac{1}{r}\right) = \partial_x \left(\frac{-x}{r^3}\right) = \frac{-(r^3 - 3x^2 r)}{r^6} = \frac{-(r^2 - 3x^2)}{r^5}.$$

By symmetry, $\partial_{yy}(1/r) = \frac{-(r^2-3y^2)}{r^5}$, $\partial_{zz}(1/r) = \frac{-(r^2-3z^2)}{r^5}$. Summing:

$$\nabla^2 \left(\frac{1}{r}\right) = \frac{-[(r^2 - 3x^2) + (r^2 - 3y^2) + (r^2 - 3z^2)]}{r^5} = \frac{-[3r^2 - 3r^2]}{r^5} = 0 \quad (r \neq 0).$$

For the integral of $\nabla^2(1/r)$ over a volume V containing the origin: since the classical Laplacian vanishes except at the origin, any nonzero contribution must come from the singularity.

Green's theorem in a punctured domain

Recall Green's theorem (Green's second identity). Apply it to a smooth test function $\Phi(\underline{r})$ with compact support (or sufficient decay) in the punctured domain $\mathbb{R}^3 \setminus B_\epsilon(0)$, where $B_\epsilon(0)$ is a small ball of radius ϵ centered at the origin.

$$\int_{\mathbb{R}^3} \nabla^2 \left(\frac{1}{r}\right) \Phi(\underline{r})\, \mathrm{d}^3 r = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^3 \setminus B_\epsilon(0)} \nabla^2 \left(\frac{1}{r}\right) \Phi(\underline{r})\, \mathrm{d}^3 r.$$

Green's theorem:

$$\int_V \left[\Phi \nabla^2 \Psi - \Psi \nabla^2 \Phi\right] \mathrm{d}^3 x = \oint_S \left[\Phi \frac{\partial \Psi}{\partial n} - \Psi \frac{\partial \Phi}{\partial n}\right] \mathrm{d}a.$$

Since $\nabla^2(1/r) = 0$ in the punctured domain, take $\Psi = 1/r$, $\Phi = \Phi(\underline{r})$:

$$\int_{|\underline{r}|>\epsilon} \left[\frac{1}{r} \nabla^2 \Phi - \Phi\, \nabla^2 \left(\frac{1}{r}\right)\right] \mathrm{d}^3 r = \oint_{\partial(|\underline{r}|>\epsilon)} \left(\frac{1}{r} \nabla \Phi - \Phi\, \nabla \left(\frac{1}{r}\right)\right) \cdot \mathrm{d}\underline{A}.$$

The outer boundary at ∞ contributes zero (Φ and $1/r$ decay). The inner boundary is the sphere S_ϵ of radius ϵ. For the exterior domain $|\underline{r}| > \epsilon$, the outward normal points inward (toward the origin), so $\mathrm{d}\underline{A} = -\hat{\underline{r}}\, \epsilon^2 \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi$.

Note: The boundary of the punctured domain consists of the outer boundary at ∞ (no contribution) and the inner boundary $S_\epsilon = \partial B_\epsilon$. On S_ϵ, the outward normal w.r.t. the exterior is $-\hat{\underline{r}}$ (anti-radial).

*The sphere is of fixed radius ϵ s.t. ρ changes little over its interior. When regularized kernels are used, $a \ll \epsilon$ (or later R), and $a \to 0$ in the limit; this justifies considering only the contribution from inside the sphere.

Evaluation of the surface integral as $\epsilon \to 0$

Since $\nabla^2(1/r) = 0$ in the punctured region,

$$\int_{|\underline{r}|>\epsilon} \frac{1}{r}\,\nabla^2\Phi\,\mathrm{d}^3r = -\oint_{S_\epsilon} \left(\frac{1}{r}\,\frac{\partial\Phi}{\partial n} - \Phi\,\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right)\mathrm{d}A \tag{*}$$

where on S_ϵ, $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$.

As $\epsilon \to 0$, Φ and $\nabla\Phi$ are essentially constant on the small sphere, equal to $\Phi(0)$ and $\nabla\Phi(0)$.

First term of RHS. On S_ϵ, $r = \epsilon$ so $1/r = 1/\epsilon$. Then

$$\oint_{S_\epsilon} \frac{1}{r}\,\frac{\partial\Phi}{\partial n}\,\mathrm{d}A = \frac{1}{\epsilon}\oint_{S_\epsilon} \frac{\partial\Phi}{\partial n}\,\mathrm{d}A.$$

Since Φ is C^2, expand $\Phi(\underline{r}) = \Phi(0) + \nabla\Phi(0)\cdot\underline{r} + \mathcal{O}(r^2)$.
On S_ϵ, $(\underline{r} = \epsilon\hat{\underline{r}})$ thus,

$$\frac{\partial\Phi}{\partial r} = \nabla\Phi\cdot\hat{\underline{r}} = \nabla\Phi(0)\cdot\hat{\underline{r}} + \mathcal{O}(\epsilon).$$

Integrate over the sphere (surface area $\approx 4\pi\epsilon^2$): by symmetry,

$$\oint_{S_\epsilon} (\underline{N}\cdot\hat{\underline{r}})\,\mathrm{d}A = \underline{N}\cdot\oint \hat{\underline{r}}\,\mathrm{d}A = 0$$

(integral of $\hat{\underline{r}}$ over sphere is zero). So $\oint_{S_\epsilon} \frac{\partial\Phi}{\partial r}\,\mathrm{d}A = \mathcal{O}(\epsilon^3)$, hence $\oint_{S_\epsilon} \frac{1}{\epsilon}\,\frac{\partial\Phi}{\partial n}\,\mathrm{d}A = \mathcal{O}(\epsilon^2) \to 0$.

Second term of RHS of (*).

$$\oint_{S_\epsilon} \Phi\,\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\mathrm{d}A = \oint_{S_\epsilon} \Phi\left(-\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right)\epsilon^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \oint_{S_\epsilon} \Phi\,\frac{1}{r^2}\,\mathrm{d}A,$$

since $\frac{\partial}{\partial r}(1/r) = -1/r^2$.

Dirac delta and regularized potential

On the sphere $r = \epsilon$, so

$$\oint_{S_\epsilon} \frac{\Phi}{\epsilon^2}\,\mathrm{d}A = \Phi(0)\cdot 4\pi\epsilon^2/\epsilon^2 + \text{const.} = 4\pi\Phi(0) + \text{const.}$$

Taking $\epsilon \to 0$: $\int_{\mathbb{R}^3} \frac{1}{r}\,\nabla^2\Phi\,\mathrm{d}^3r = 4\pi\Phi(0)$. Integration by parts gives

$$\int_{\mathbb{R}^3} \nabla^2\left(\frac{1}{r}\right)\Phi\,\mathrm{d}^3r = -4\pi\Phi(0) = \int_{\mathbb{R}^3} (-4\pi\,\delta^3(\underline{r}))\Phi(\underline{r})\,\mathrm{d}^3r$$

$\Rightarrow \nabla^2(1/r) = -4\pi\,\delta^3(\underline{r})$ (with $r = |\underline{r}|$; δ^3 the 3D Dirac delta at the origin).

Recall: $\delta(x-a) = 0$ for $x \neq a$; $\int \delta(x-a)\,\mathrm{d}x = 1$ if the region includes $x = a$, else 0; and $\delta(f(x)) = \sum_i \frac{1}{|\frac{\partial f}{\partial x}|}\,\delta(x - x_i)$ at roots x_i of f.

Thus we have a formal statement of the singular nature of $\nabla^2(1/r)$.

Recall the family of regularized kernels $G_a(r) = \frac{1}{\sqrt{r^2+a^2}}$. Define the **regularized potential**:

$$\boxed{\Phi_a(\underline{x}) = \frac{1}{4\pi\epsilon_0} \int_{\mathbb{R}^3} \rho(\underline{x}')\, G_a(|\underline{x} - \underline{x}'|)\, \mathrm{d}^3 x'.}$$

Consider the Laplacian of the regularized potential (as in the non-regularized case):

$$\nabla^2 \Phi_a(\underline{x}) = \frac{1}{4\pi\epsilon_0} \int_{\mathbb{R}^3} \rho(\underline{x}')\, \nabla^2 G_a(|\underline{x} - \underline{x}'|)\, \mathrm{d}^3 x'.$$

Dominated convergence (smoothness of G_a, derivatives decay sufficiently at ∞, ρ with compact support or suitable decay) justifies interchanging the Laplacian and integral. Choose spherical coordinates with origin at $\underline{x}$ so that $r = |\underline{x} - \underline{x}'|$. For a radial function, $\nabla^2 G_a(r) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial G_a}{\partial r} \right)$. Compute:

$$\frac{\partial G_a}{\partial r} = \frac{-r}{(r^2 + a^2)^{3/2}} \Rightarrow r^2 \frac{\partial G_a}{\partial r} = \frac{-r^3}{(r^2 + a^2)^{3/2}}.$$

Quotient rule and Taylor expansion

To compute $\frac{\partial}{\partial r} \left(\frac{-r^3}{(r^2+a^2)^{3/2}} \right)$, use the quotient rule.
Let $u = r^3$, $v = (r^2 + a^2)^{3/2}$; then

$$\frac{\partial}{\partial r}(-u/v) = -\frac{u'v - uv'}{v^2}.$$

We have $u' = 3r^2$, $v' = \frac{3}{2}(r^2 + a^2)^{1/2} \cdot 2r = 3r(r^2 + a^2)^{1/2}$. So

$$u'v - uv' = 3r^2(r^2 + a^2)^{3/2} - r^3 \cdot 3r(r^2 + a^2)^{1/2} = (r^2 + a^2)^{1/2} \left[3r^2(r^2 + a^2) - 3r^4 \right] = 3(r^2 + a^2)^{1/2}\, a^2 r^2.$$

Hence

$$\frac{\partial}{\partial r} \left(\frac{-r^3}{(r^2 + a^2)^{3/2}} \right) = -\frac{3a^2 r^2}{(r^2 + a^2)^{5/2}}, \qquad \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial G_a}{\partial r} \right) = -\frac{3a^2}{(r^2 + a^2)^{5/2}}.$$

Thus

$$\nabla^2 \Phi_a(\underline{x}) = -\frac{1}{4\pi\epsilon_0} \int \rho(\underline{x}')\, \frac{3a^2}{(r^2 + a^2)^{5/2}}\, \mathrm{d}^3 x', \qquad r = |\underline{x} - \underline{x}'|.$$

Change of variables: $s = \underline{x}' - \underline{x} \Rightarrow \mathrm{d}^3 x' = \mathrm{d}^3 s$, $\underline{x}' = \underline{x} + \underline{s}$, $r = |\underline{s}|$. Write $|\underline{s}| = s$ for simplicity. Then

$$\nabla^2 \Phi_a(\underline{x}) = -\frac{3a^2}{4\pi\epsilon_0} \int_{\mathbb{R}^3} \frac{\rho(\underline{x} + \underline{s})}{(s^2 + a^2)^{5/2}}\, \mathrm{d}^3 s.$$

Expand $\rho(\underline{x} + \underline{s}) = \rho(\underline{x}) + \nabla\rho(\underline{x}) \cdot \underline{s} + \frac{1}{2}\, s_i s_j\, \partial_i \partial_j \rho(\underline{x}) + \mathcal{O}(s^3)$
(remainder $|R_2(\underline{x}; \underline{s})| \le C|s|^3$ for const. C depending on third derivatives of ρ near $\underline{x}$).
Insert into the integral and decompose:

$$\int_{\mathbb{R}^3} \frac{\rho(\underline{x} + \underline{s})}{(s^2 + a^2)^{5/2}}\, \mathrm{d}^3 s = \rho(\underline{x})\, A(a) + \partial_k \rho(\underline{x})\, B_k(a) + \frac{1}{2}\, \partial_i \partial_j \rho(\underline{x})\, C_{ij}(a) + \int_{\mathbb{R}^3} \frac{R_2(\underline{x}; \underline{s})}{(s^2 + a^2)^{5/2}}\, \mathrm{d}^3 s,$$

where

$$A(a) := \int_{\mathbb{R}^3} \frac{1}{(s^2 + a^2)^{5/2}}\, \mathrm{d}^3 s,$$
$$B_k(a) := \int_{\mathbb{R}^3} \frac{s_k}{(s^2 + a^2)^{5/2}}\, \mathrm{d}^3 s \quad (k = 1, 2, 3),$$
$$C_{ij}(a) := \int_{\mathbb{R}^3} \frac{s_i s_j}{(s^2 + a^2)^{5/2}}\, \mathrm{d}^3 s \quad (i, j = 1, 2, 3).$$

Computation of $A(a)$

Compute

$$I := \int_0^\infty \frac{s^2}{(s^2+a^2)^{5/2}}\,\mathrm{d}s.$$

Substitute $s = at$, $\mathrm{d}s = a\,\mathrm{d}t$:

$$I = \int_0^\infty \frac{a^2t^2}{a^5(t^2+1)^{5/2}}\,a\,\mathrm{d}t = \frac{1}{a^2}\int_0^\infty \frac{t^2}{(1+t^2)^{5/2}}\,\mathrm{d}t = \frac{1}{a^2}\,J.$$

To evaluate J, use $t = \tan\theta$, $\mathrm{d}t = \sec^2\theta\,\mathrm{d}\theta$, $\theta \in [0, \pi/2)$. Then

$$\frac{t^2}{(1+t^2)^{5/2}}\,\mathrm{d}t = \frac{\tan^2\theta}{\sec^5\theta}\,\sec^2\theta\,\mathrm{d}\theta = \tan^2\theta\,\cos^3\theta\,\mathrm{d}\theta = \sin^2\theta\,\cos\theta\,\mathrm{d}\theta.$$

So

$$J = \int_0^{\pi/2} \sin^2\theta\,\cos\theta\,\mathrm{d}\theta.$$

Let $u = \sin\theta$, $\mathrm{d}u = \cos\theta\,\mathrm{d}\theta$; $\theta = 0 \Rightarrow u = 0$, $\theta = \pi/2 \Rightarrow u = 1$. Hence $J = \int_0^1 u^2\,\mathrm{d}u = \frac{1}{3}$. Thus $I = \frac{1}{3a^2}$.

In 3D,

$$A(a) = 4\pi\int_0^\infty \frac{s^2}{(s^2+a^2)^{5/2}}\,\mathrm{d}s = 4\pi I = \frac{4\pi}{3a^2}.$$

Evaluation of $A(a)$ and $B_k(a)$

$A(a)$ in spherical coordinates

Evaluate

$$A(a) = \int_{\mathbb{R}^3} \frac{1}{(s^2+a^2)^{5/2}}\,\mathrm{d}^3s$$

where $s = |\underline{r}|$ and $a > 0$.
Since the integrand depends only on s, use $\mathrm{d}^3s = s^2\,\mathrm{d}s\,\mathrm{d}\Omega$ and $\int \mathrm{d}\Omega = 4\pi$, so

$$A(a) = 4\pi\int_0^\infty \frac{s^2}{(s^2+a^2)^{5/2}}\,\mathrm{d}s.$$

By symmetry over $\mathbb{R}^3$, the integrand of $B_k(a)$ is $\frac{s_k}{(s^2+a^2)^{5/2}}$, which is odd in s_k while the rest is even $\Rightarrow$ $B_k(a) = 0$ for all k.

Isotropic tensor $C_{ij}(a)$ and quadratic term

After Taylor expanding $\rho(\underline{x}+\underline{s})$ to second order, the quadratic contribution to the integral is

$$\frac{1}{2}(\partial_i\partial_j\rho)(\underline{x})\int_{\mathbb{R}^3} \frac{s_is_j}{(s^2+a^2)^{5/2}}\,\mathrm{d}^3s = \frac{1}{2}(\partial_i\partial_j\rho)(\underline{x})\,C_{ij}(a),$$

where $C_{ij}(a)$ is the symmetric tensor defined by this integral ($C_{ji} = C_{ij}$). The integrand and domain are rotationally invariant $\Rightarrow$ $C_{ij}(a)$ is an isotropic tensor of rank 2. In 3D the only such (from the metric) is a multiple of δ_{ij}:

$$\boxed{C_{ij}(a) = \lambda(a)\,\delta_{ij} \quad \text{for some scalar } \lambda(a).}$$

Contract with δ_{ij} (trace):

$$C_{ii}(a) = \int_{\mathbb{R}^3} \frac{s^2}{(s^2+a^2)^{5/2}}\,\mathrm{d}^3s = 3\lambda(a) \Rightarrow \lambda(a) = \frac{1}{3}\int_{\mathbb{R}^3} \frac{s^2}{(s^2+a^2)^{5/2}}\,\mathrm{d}^3s \propto \mathcal{O}(1/a^2).$$

The full quadratic term is $\frac{1}{2}(\partial_i\partial_j\rho)\,C_{ij} = \frac{1}{2}\lambda(a)\,\nabla^2\rho(\underline{x})$ (since $(\partial_i\partial_j\rho)\delta_{ij} = \nabla^2\rho$). The remainder term converges and scales as $\mathcal{O}(a^2)$ as $a \to 0$.

Poisson equation and boundary conditions

Limit $a \to 0$ and surface charge

Thus as $a \to 0$,

$$\int \frac{\rho(\underline{x}+\underline{s})}{(s^2+a^2)^{5/2}}\,\mathrm{d}^3 s = \rho(\underline{x})\,\frac{4\pi}{3a^2} + \mathcal{O}(1/a) + \mathcal{O}(a^0),$$

$$\nabla^2\Phi_a = -\frac{3a^2}{4\pi\epsilon_0}\left(\rho\,\frac{4\pi}{3a^2} + \mathcal{O}(1/a)\right) = -\frac{\rho}{\epsilon_0} + \mathcal{O}(a) \Rightarrow \lim_{a\to 0^+} \nabla^2\Phi_a = -\frac{\rho}{\epsilon_0}.$$

Poisson's equation.

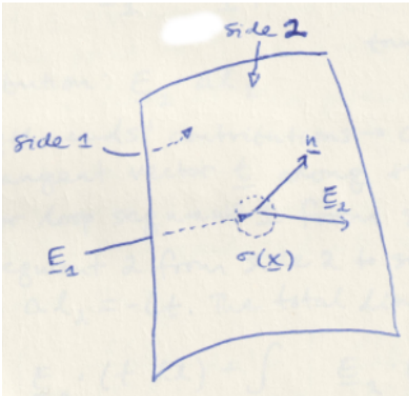

Figure 3: Discontinuity in the normal component of the electric field across a surface layer of charge. $\underline{n}$ is directed from side 1 to side 2 of the surface. $\sigma(\underline{x}) :=$ surface-charge density (C/m²).

Gauss's Law $\Rightarrow (\underline{E}_2 - \underline{E}_1)\cdot\underline{n} = \sigma/\epsilon_0$, i.e., a jump σ/ϵ_0 in the normal component when crossing the surface in the direction of $\underline{n}$.

Tangential continuity of $\underline{E}$

Show the tangential component of $\underline{E}$ is continuous across a surface with surface charge σ, using $\oint \underline{E}\cdot\mathrm{d}\underline{l} = 0$.

Consider a thin rectangular loop straddling the boundary; as the short ends shrink ($\epsilon \to 0$), only the long sides (length L) contribute.

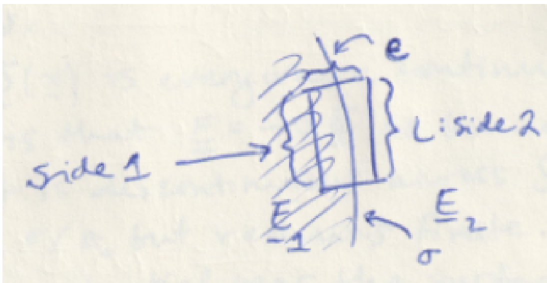

Figure 4: As $\epsilon \to 0$, the loop encloses no net charge and the path becomes infinitesimally thin while still crossing the boundary. Rectangular loop (Side 1, Side 2, length L) with field $\underline{E}_1$, $\underline{E}_2$; σ is the surface charge density.

With unit tangent $\hat{\underline{t}}$, $\Delta\underline{l}_1 = L\hat{\underline{t}}$ and $\Delta\underline{l}_2 = -L\hat{\underline{t}}$. So $\oint \underline{E}\cdot\mathrm{d}\underline{l} = E_{t,1}L - E_{t,2}L = 0 \Rightarrow E_{t,1} = E_{t,2}$. Hence $(\underline{E}_2 - \underline{E}_1)\cdot\hat{\underline{t}} = 0$ for any tangential $\hat{\underline{t}} \Rightarrow$ tangential component of $\underline{E}$ is continuous.

Potential continuity

Potential at any point; splitting the integral

Potential at any point:

$$\Phi(\underline{x}) = \frac{1}{4\pi\epsilon_0}\int_S \frac{\sigma(\underline{x}')}{|\underline{x}-\underline{x}'|}\,\mathrm{d}a.'$$

For volume or surface charge, Φ is continuous; for point/line charges or dipole layers it need not be.
Key: $\underline{E} = -\nabla\Phi$ is bounded (finite) everywhere (normal component jumps by σ/ϵ_0 but stays finite).
To show Φ continuous:
1) Split the potential near the surface: small disk D_δ on S centered at $\underline{x}_0$ with radius δ, remainder $S \setminus D_\delta$. So $\Phi(\underline{x}) = I_1(\underline{x}) + I_2(\underline{x})$.
2) The "distant" part I_1: for $\underline{x}' \in S \setminus D_\delta$, $|\underline{x}-\underline{x}'|$ is bounded away from zero ($\geq \delta$), σ bounded $\Rightarrow I_1(\underline{x})$ continuous at $\underline{x}_0$, and

$$\lim_{\underline{x}\to\underline{x}_0} I_1(\underline{x}) = \frac{1}{4\pi\epsilon_0}\int_{S\setminus D_\delta} \frac{\sigma(\underline{x}')}{|\underline{x}_0-\underline{x}'|}\,\mathrm{d}a'.$$

Bounding I_2; continuity via bounded $\underline{E}$

3) Bound the "near" singular part I_2: use local flatness of S, signed distance h along $\underline{n}$ at $\underline{x}_0$, $\underline{x} = \underline{x}_0 + h\underline{n}$. In the local disk D_δ, $|\underline{x}-\underline{x}'| = \sqrt{\rho^2+h^2}$, $\mathrm{d}a' = \rho\,\mathrm{d}\rho\,\mathrm{d}\phi$.
If $|\sigma| \leq M$

$$\begin{aligned}
\Rightarrow |I_2(\underline{x})| &\leq \frac{M}{4\pi\epsilon_0}\int_0^\delta\int_0^{2\pi} \frac{\rho\,\mathrm{d}\rho\,\mathrm{d}\phi}{\sqrt{\rho^2+h^2}} \\
&= \frac{M}{2\epsilon_0}\int_0^\delta \frac{\rho\,\mathrm{d}\rho}{\sqrt{\rho^2+h^2}} \\
&= \frac{M}{2\epsilon_0}(\sqrt{\delta^2+h^2} - |h|) \leq \frac{M\delta}{2\epsilon_0}.
\end{aligned}$$

So I_2 is bounded; with bounded $\underline{E}$, the fundamental theorem gives

$$|\Phi(\underline{x}_B) - \Phi(\underline{x}_A)| = \left|\int_{\underline{x}_A}^{\underline{x}_B} \underline{E}\cdot\mathrm{d}\underline{l}\right| \leq \sup|E_n|\,|h_B - h_A|$$

along the normal $\Rightarrow \Phi$ has a unique limit as $\underline{x} \to \underline{x}_0$ from either side.

Direct proof via bounded $\underline{E}$

Prove $\Phi(\underline{x}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\underline{x}')}{|\underline{x}-\underline{x}'|}\,\mathrm{d}V'$ is continuous everywhere in $\mathbb{R}^3$ (even inside V), assuming ρ bounded and V of finite extent or ρ decaying.
Key: $\underline{E} = -\nabla\Phi$ is bounded everywhere.
For any two points $\underline{x}_A$, $\underline{x}_B$, $\Phi(\underline{x}_B) - \Phi(\underline{x}_A) = -\int_{\underline{x}_A}^{\underline{x}_B} \underline{E}\cdot\mathrm{d}\underline{l}$.
Along a straight path, $|\Phi(\underline{x}_B) - \Phi(\underline{x}_A)| \leq \|\underline{E}\|_\infty^{\text{seg}}|\underline{x}_B - \underline{x}_A|$;
$\underline{E}$ bounded on compact sets $\Rightarrow \Phi$ is (locally) Lipschitz, hence continuous.
So $\lim_{\underline{x}\to\underline{x}_0}\Phi(\underline{x}) = \Phi(\underline{x}_0)$.

Splitting the integral to show bounded $\underline{E}$

2) Direct proof: split near $\underline{x}_0$: $\Phi(\underline{x}) = J_1 + J_2$ where J_1 is the integral over $|\underline{x}' - \underline{x}_0| > \delta$ and J_2 over $|\underline{x}' - \underline{x}_0| \leq \delta$.
J_1 is continuous at $\underline{x}_0$.
For J_2, $|\rho| \leq M$; use spherical coords centered at

$$\underline{x}_0,\ \underline{x} = \underline{x}_0 + \underline{h},\ |J_2| \leq \frac{M}{4\pi\epsilon_0} \int_{|\underline{r}'|\leq\delta} \frac{\mathrm{d}V'}{|\underline{h} - \underline{r}'|}.$$

For $|\underline{h}| < \delta/2$ this is bounded; maximum at $\underline{h} = 0$:

$$\int_{|\underline{r}'|\leq\delta} \frac{\mathrm{d}V'}{|\underline{r}'|} = 2\pi\delta^2 < \infty.$$

So Φ and $\nabla\Phi$ (field from the local ball) remain bounded; $\nabla(1/r)$ is integrable over a ball excluding the origin.

Green's theorem (2D) and identities

Electrostatics and Green's theorem (2D)

Most electrostatics problems involve finite regions with prescribed boundary conditions $\Rightarrow$ Laplace/Poisson solutions. The formula $\Phi(\underline{x}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\underline{x}')}{|\underline{x} - \underline{x}'|} \mathrm{d}^3x'$ is often inadequate; techniques using Green's theorem are used.
Green's theorem: line integral around a simple closed curve C equals a double integral over the plane region D bounded by C (2D case of Stokes).
In 1D it is the fundamental theorem of calculus; in 2D, equivalent to the divergence theorem.
Let C be positively oriented, piecewise smooth, simple closed; D the region bounded by C. If L, M are functions of (x, y) on an open region containing D with continuous partials on D, then

$$\oint_C (L\,\mathrm{d}x + M\,\mathrm{d}y) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) \mathrm{d}A,$$

path along C counter-clockwise. Used for area and centroid of plane figures.

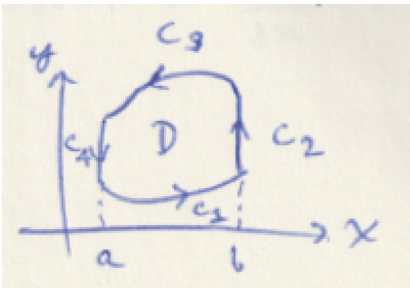

Figure 5: Region D bounded by curves c_1, c_2, c_3, c_4 with counter-clockwise orientation; x-axis bounds at a and b.

Singular nature of Laplacian; identity $\nabla \cdot (\phi \nabla \psi)$

Scalar potential as solution of Laplace's equation; singular nature of

$$\nabla^2(1/|\underline{x} - \underline{x}'|) = -4\pi\delta(\underline{x} - \underline{x}').$$

Divergence theorem and Green's identities convert the Poisson PDE into an integral equation by choosing $\psi = 1/R = 1/|\underline{x} - \underline{x}'|$; appropriate boundary conditions give unique solution.
Let $\underline{A} = \phi\nabla\psi$. Verify that

$$\nabla \cdot \underline{A} = \nabla \cdot (\phi \nabla \psi) = \phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi$$

in rectangular coordinates:

$$\nabla = \hat{\underline{x}}\partial_x + \hat{\underline{y}}\partial_y + \hat{\underline{z}}\partial_z,$$

product rule on each component, then

$$\phi\nabla^2\psi + \nabla\phi \cdot \nabla\psi.$$

By definition of normal derivative,

$$\frac{\partial \psi}{\partial n} = \nabla\psi \cdot \underline{n} \Rightarrow \phi\nabla\psi \cdot \underline{n} = \phi\,\frac{\partial \psi}{\partial n}.$$

Green's first and second identities; application

Divergence theorem:

$$\int_U \nabla \cdot \underline{u}\,\mathrm{d}^3x = \int_{\partial U} \underline{u} \cdot \underline{n}\,\mathrm{d}S \text{ (unit normal } \underline{n}\text{), for } \underline{u} \in C^1(\overline{U})\text{; } U \text{ bounded open, } \partial U\ C^1.$$

Green's formulas:

$$D := \nabla,\ \ \Delta := \nabla^2$$

$$\int_U \Delta u\,\mathrm{d}^3x = \int_{\partial U} \frac{\partial u}{\partial n}\,\mathrm{d}S.$$

Green's first identity:

$$\int_U \nabla v \cdot \nabla u\,\mathrm{d}^3x = -\int_U u\,\Delta v\,\mathrm{d}^3x + \int_{\partial U} u\,\frac{\partial v}{\partial n}\,\mathrm{d}S.$$

Green's second:

$$\int_U (u\Delta v - v\Delta u)\,\mathrm{d}^3x = \int_{\partial U} \left(u\,\frac{\partial v}{\partial n} - v\,\frac{\partial u}{\partial n}\right)\mathrm{d}S.$$

Substitute $\nabla \cdot (\Phi\nabla\Psi)$ and $\nabla\Psi \cdot \underline{n}$ into the divergence theorem

$$\Rightarrow \int_V (\Phi\nabla^2\Psi + \nabla\Phi \cdot \nabla\Psi)\,\mathrm{d}^3x = \oint_S \Phi\,\frac{\partial \Psi}{\partial n}\,\mathrm{d}a. \tag{1}$$

Interchange Φ and Ψ to get (2); (1)$-$(2) and $\nabla\Phi \cdot \nabla\Psi = \nabla\Psi \cdot \nabla\Phi$ give

$$\int_V (\Phi\nabla^2\Psi - \Psi\nabla^2\Phi)\,\mathrm{d}^3x = \oint_S [\Phi\,\frac{\partial \Psi}{\partial n} - \Psi\,\frac{\partial \Phi}{\partial n}]\,\mathrm{d}a \qquad \text{(Green's 2nd).}$$

For electrostatics, $\gamma = 1/R = 1/|\underline{x} - \underline{x}'|$, Poisson $\nabla^2\Phi = -\rho/\epsilon_0$, $\nabla^2(1/|\underline{x} - \underline{x}'|) = -4\pi\delta(\underline{x} - \underline{x}')$; rewrite Green's 2nd to get the integral relation involving Φ, ρ, and surface terms.

Green's integral solution for Φ

$$\int_V \left[-4\pi\Phi(\underline{x}')\delta(\underline{x}-\underline{x}') + \frac{1}{\epsilon_0 R}\,\rho(\underline{x}')\right] \mathrm{d}^3x' = \oint_S \left[\Phi\,\frac{\partial}{\partial n'}\left(\frac{1}{R}\right) - \frac{1}{R}\,\frac{\partial \Phi}{\partial n'}\right] \mathrm{d}a'.$$

Identity: $\int f(\underline{x})\,\delta(\underline{x}-\underline{a})\,\mathrm{d}^3x = f(\underline{a})$. So

$$\Phi(\underline{x}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\underline{x}')}{R}\,\mathrm{d}^3x' + \frac{1}{4\pi}\oint_S \left[\frac{1}{R}\frac{\partial\Phi}{\partial n'} - \Phi\,\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right] \mathrm{d}a' \qquad (*)$$

(volume integral $= 0$ if $\underline{x}$ outside S).
1) If $S \to \infty$ and $\underline{E}$ on S falls off faster than R^{-2}, the surface integral vanishes

$$\Rightarrow \Phi = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\underline{x}')}{|\underline{x}-\underline{x}'|}\,\mathrm{d}^3x.$$

2) For a charge-free volume, (*) expresses Φ (solution of Laplace) in terms of Φ and $\partial\Phi/\partial n$ on the boundary—an integral statement, not a boundary-value solution; Cauchy (both Φ and $\partial\Phi/\partial n$) overspecifies the problem.

Uniqueness for Poisson with Dirichlet or Neumann

Show uniqueness of the solution of Poisson's equation $\nabla^2\Phi = -\rho/\epsilon_0$ in V with Dirichlet or Neumann conditions on S.
By contradiction: suppose Φ_1, Φ_2 both satisfy the same b.c.
Let $U = \Phi_2 - \Phi_1$; then $\nabla^2 U = 0$, and $U = 0$ (Dirichlet) or $\frac{\partial U}{\partial n} = 0$ (Neumann) on S.
Green's identity with $\phi = \psi = U$:

$$\int_V (U\nabla^2 U + |\nabla U|^2)\,\mathrm{d}^3x = \oint_S U\,\frac{\partial U}{\partial n}\,\mathrm{d}a \qquad (*)$$

Dirichlet $\Rightarrow$

$$\int_V |\nabla U|^2\,\mathrm{d}^3x = 0 \Rightarrow \nabla U = 0$$

(since $|\nabla U|^2 \geq 0$), so U constant; $U = 0$ on S and V connected $\Rightarrow U = 0$ in $V \Rightarrow \Phi_1 = \Phi_2$.
Neumann $\Rightarrow$

$$\oint_S U\,\frac{\partial U}{\partial n}\,\mathrm{d}a = 0$$

$\Rightarrow$ again $\int_V |\nabla U|^2\,\mathrm{d}^3x = 0 \Rightarrow$ uniqueness.
Electrostatic problems are specified by Dirichlet or Neumann (possibly partly at ∞).
Do Cauchy conditions on an open surface define a unique electrostatic problem?

Linear differential operators and Green's function

A linear differential operator L acts on functions by linear combinations of the function and its derivatives; $L(\alpha u + \beta v) = \alpha L(u) + \beta L(v)$. Typical form:

$$L = a_n(x)\frac{\mathrm{d}^n}{\mathrm{d}x^n} + \cdots + a_0(x);$$

$L[y] = f(x)$ is nonhomogeneous, $L[y] = 0$ homogeneous.
Examples: Laplacian $\Delta = \nabla^2$ (linear); heat operator $\frac{\partial u}{\partial t} - k\Delta u$.
Non-example: $L[y] = y\,\frac{\mathrm{d}y}{\mathrm{d}x}$ is not linear. A Green's function $G(\underline{x},\underline{s})$ for L at a point $\underline{s}$ satisfies $LG(\underline{x},\underline{s}) = \delta(\underline{s}-\underline{x})$. Then the solution of $Lu = f$ is given by convolution with G (superposition).

Green's 2nd identity; exterior Neumann problem

$$\Phi(\underline{x}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\underline{x}')}{R}\,\mathrm{d}^3x' + \frac{1}{4\pi}\oint_S \left[\frac{1}{R}\frac{\partial\Phi}{\partial n'} - \Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right]\mathrm{d}a'.$$

Green's 2nd:

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,\mathrm{d}^3x = \oint_S \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right]\mathrm{d}a.$$

The customary Neumann "exterior problem" has V bounded by a finite closed surface and a surface at ∞ $\Rightarrow$ surface area infinite $\Rightarrow$ boundary condition becomes homogeneous and $\langle\Phi\rangle_S$ vanishes.

Green's functions G_D, G_N for Dirichlet and Neumann

Green's 2nd with $\psi = 1/R = 1/|\underline{x}-\underline{x}'|$, $\phi = \Phi$, and $\nabla^2(1/R) = -4\pi\delta(\underline{x}-\underline{x}')$ gives a result (not a solution by itself). Green functions $G(\underline{x},\underline{x}')$ satisfy $\nabla^2 G = -4\pi\delta(\underline{x}-\underline{x}')$, with

$$G = \frac{1}{4\pi}\frac{1}{|\underline{x}-\underline{x}'|} + F\ \nabla^2 F = 0 \text{ in } V.$$

Using $\psi = G$ in Green's theorem yields

$$\Phi(\underline{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho(\underline{x}')\,G\,\mathrm{d}^3x' + \frac{1}{4\pi}\oint_S \left[G\frac{\partial\Phi}{\partial n'} - \Phi\frac{\partial G}{\partial n'}\right]\mathrm{d}a'. \qquad (*)$$

Dirichlet: choose G_D so $G_D = 0$ on $S \Rightarrow$

$$\Phi(\underline{x}) = \frac{1}{4\pi\epsilon_0}\int_V \rho\,G_D\,\mathrm{d}^3x' - \frac{1}{4\pi}\oint_S \Phi\frac{\partial G_D}{\partial n'}\,\mathrm{d}a'.$$

Neumann: $\frac{\partial G_N}{\partial n'} = -\frac{1}{S}$ on S (since $\oint_S \frac{\partial G}{\partial n'}\,\mathrm{d}a' = -4\pi$, $\frac{\partial G_N}{\partial n'} = 0$ is not allowed).
Then

$$\Phi(\underline{x}) = \langle\Phi\rangle_S + \frac{1}{4\pi\epsilon_0}\int_V \rho\,G_N\,\mathrm{d}^3x' + \frac{1}{4\pi}\oint_S \frac{\partial\Phi}{\partial n'}G_N\,\mathrm{d}a',$$

where $\langle\Phi\rangle_S$ is the average of Φ over S.

Work and potential energy; field energy

Work to bring q_i from infinity to $\underline{x}_i$ in potential Φ:

$$W_i = q_i\Phi(\underline{x}_i);\ \Phi(\underline{x}_i) = \frac{1}{4\pi\epsilon_0}\sum_{j\neq i}\frac{q_j}{|\underline{x}_i-\underline{x}_j|} \Rightarrow W_i = \frac{q_i}{4\pi\epsilon_0}\sum_{j\neq i}\frac{q_j}{|\underline{x}_i-\underline{x}_j|}.$$

Total potential energy:

$$W = \frac{1}{4\pi\epsilon_0}\sum_i\sum_{j\neq i}\frac{q_iq_j}{|\underline{x}_i-\underline{x}_j|} = \frac{1}{8\pi\epsilon_0}\sum_{i,j,\,i\neq j}\frac{q_iq_j}{|\underline{x}_i-\underline{x}_j|}$$

(divide by 2; omit self-energy).
Continuous distribution:

$$W = \frac{1}{8\pi\epsilon_0}\iint \frac{\rho(\underline{x})\rho(\underline{x}')}{|\underline{x}-\underline{x}'|}\,\mathrm{d}^3x\,\mathrm{d}^3x'.$$

One integral is

$$(4\pi\epsilon_0)^{-1}\int \frac{\rho(\underline{x}')}{|\underline{x}-\underline{x}'|}\,\mathrm{d}^3x' = \Phi(\underline{x}) \Rightarrow W = \frac{1}{2}\int \rho(\underline{x})\Phi(\underline{x})\,\mathrm{d}^3x.$$

Using Poisson

$$\nabla^2\Phi = -\rho/\epsilon_0:\ W = -\frac{\epsilon_0}{2}\int \Phi\nabla^2\Phi\,\mathrm{d}^3x;$$

integration by parts $\Rightarrow$

$$W = \frac{\epsilon_0}{2}\int |\nabla\Phi|^2\,\mathrm{d}^3x = \frac{\epsilon_0}{2}\int |\underline{E}|^2\,\mathrm{d}^3x.$$

Integrand $w = \frac{\epsilon_0}{2}|\underline{E}|^2$ is energy density.

Self-energy vs interaction; W_{int}

$w = \frac{\epsilon_0}{2}|\underline{E}|^2$ is positive-definite; its volume integral is ≥ 0. This seems to contradict $W = \frac{1}{8\pi\epsilon_0}\sum_{i,j}\frac{q_i q_j}{|\underline{x}_i - \underline{x}_j|}$ (e.g. two opposite charges give negative PE). The field energy includes *self-energy* contributions; the sum formula does not.

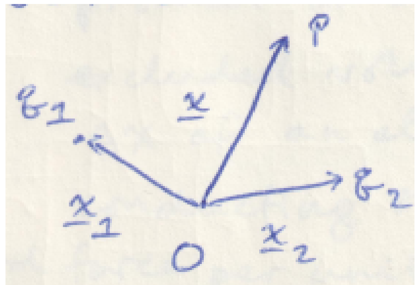

Figure 6: Two point charges q_1, q_2 at $\underline{x}_1$, $\underline{x}_2$: field at P (position $\underline{x}$); self-energy and interaction contributions to energy density w.

For two point charges q_1, q_2 at $\underline{x}_1$, $\underline{x}_2$,

$$\underline{E} = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1(\underline{x}-\underline{x}_1)}{|\underline{x}-\underline{x}_1|^3} + \frac{q_2(\underline{x}-\underline{x}_2)}{|\underline{x}-\underline{x}_2|^3}\right).$$

So $32\pi^2\epsilon_0 w$ has terms

$$\frac{q_1^2}{|\underline{x}-\underline{x}_1|^4} + \frac{q_2^2}{|\underline{x}-\underline{x}_2|^4}\ \text{(self-energy)}\ +\ 2\,\frac{q_1q_2(\underline{x}-\underline{x}_1)\cdot(\underline{x}-\underline{x}_2)}{|\underline{x}-\underline{x}_1|^3|\underline{x}-\underline{x}_2|^3}\ \text{(interaction)}.$$

Integrating the interaction term over all space: change of variable $\underline{\rho} = (\underline{x}-\underline{x}_1)/|\underline{x}_1-\underline{x}_2|$, $\underline{n}$ unit vector along $\underline{x}_1-\underline{x}_2$; using

$$\frac{\underline{\rho}+\underline{n}}{|\underline{\rho}+\underline{n}|^3} = -\nabla_{\underline{\rho}}(1/|\underline{\rho}+\underline{n}|)$$

the dimensionless integral gives

$$4\pi \Rightarrow W_{\text{int}} = \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{|\underline{x}_1-\underline{x}_2|}.$$

Capacitance

Force from energy

Forces between charged bodies can be found from the change in total electrostatic energy under small virtual displacements; express energy so that what varies with configuration is explicit. E.g., force per unit area on a conductor with surface charge $\sigma(\underline{x})$.
Energy density near the surface: $w = \frac{1}{2}\epsilon_0 |\underline{E}|^2 = \frac{\sigma^2}{2\epsilon_0}$.
Decrease in energy when an element Δa is displaced by Δx is

$$\Delta W = -\frac{\sigma^2\,\Delta a\,\Delta x}{2\epsilon_0}$$

$$\Rightarrow \text{ Outward force per unit area at the surface } = \frac{\sigma^2}{2\epsilon_0} = w$$

(Same result from σ times the field, excluding the field of the surface element itself.)

System of conductors; coefficients of capacity

For n conductors at potentials V_i with total charges Q_i $(i = 1, \ldots, n)$ in otherwise empty space, the electrostatic PE can be written in terms of potentials and geometric coefficients of capacity.

$$V_i = \sum_{j=1}^{n} P_{ij} Q_j \text{ (potential of } i\text{th conductor; } P_{ij} \text{ depends on geometry).}$$

Inversion:

$$Q_i = \sum_{j=1}^{n} C_{ij} V_j.$$

C_{ii} = "capacities" (capacitances); C_{ij} $(i \neq j)$ = coefficients of induction.
The **capacitance** of a conductor is the charge on it when held at unit potential with all others at zero. For two conductors with equal and opposite charges in the presence of grounded conductors,
capacitance = charge on one / potential difference; express in terms of C_{ij} using $Q_i = \sum_j C_{ij} V_j$.
PE:

$$W = \frac{1}{2}\sum_i Q_i V_i = \frac{1}{2}\sum_{i,j} C_{ij} V_i V_j.$$

Expressing energy via V_i, C_{ij} or Q_i, P_{ij} allows variational methods for approximate capacitances.
Note: Error estimates for the lower bound require a Green function satisfying Dirichlet boundary conditions.